A geometric interpretation of certain sums

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Elementary school teachers frequently complain that mathematics is abstract and makes too extensive use of symbols. Geometry can help reduce this dependence on symbols. We shall use a story about a famous mathematician to illustrate how some abstractions can be formulated in terms of geometric figures.

Karl Friedrich Gauss (1777–1855) is credited with using an idea, at an early age, that is now used to find an answer to such sums as $1 + 2 + 3 + 4 + \ldots + 100$. Such a sequence is called an arithmetic progression.

As is frequently the case with bright, inquisitive children, Gauss was troublesome for his teachers. On one occasion, his teacher assigned the following addition problem, believing that it would keep the young Gauss occupied for a considerable period of time:

> *Find the sum of the counting numbers from one to one hundred inclusive.*

We could approach this problem in an obvious way, as shown in figure 1. This is a tedious method, and an elementary school child is likely to make an error. Purportedly, Gauss did not use this method. He gave the correct answer after just a moment's thought. His method may be visualized as shown in figure 2.

To use Gauss's method, we write the numbers to be added twice, once in increasing order and once in decreasing order. Reading across, we see that each pair has a sum of 101 and that there are 100 such pairs. The sum of the right-hand column, then, is found by the product 100 times 101, or 10,100. This sum, however, is twice the desired sum, since the right-hand column represents adding each number twice.

$$\begin{aligned}
1 &= 1\\
1+2 &= 3\\
(1+2)+3 &= 3+3=6\\
(1+2+3)+4 &= 6+4=10\\
(1+2+3+4)+5 &= 10+5=15\\
(1+2+3+4+5)+6 &= 15+6=21\\
& 21+7=28\\
&\vdots\\
& 4950+100=5050
\end{aligned}$$

Fig. 1

$$\begin{array}{rcrcl}
1 & + & 100 & = & 101\\
2 & + & 99 & = & 101\\
3 & + & 98 & = & 101\\
4 & + & 97 & = & \cdot\\
5 & + & 96 & = & \cdot\\
6 & + & 95 & = & \cdot\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
100 & + & 1 & = & 101
\end{array}$$

Fig. 2

The answer to the original problem, then, is

$$10{,}100 \div 2 = 5{,}050.$$

The most important idea here is that it isn't really necessary to write all the sums down; it's only necessary to think about them, since we know that the number of pairs is the same as the original number of addends and that for each pair the sum is 101.

To see that the idea is a workable one, consider all the counting numbers from 1 to 10,000 and ask what is their sum. Again, pairing each number with its opposite (in the sense of reversed counting), we would have 10,000 pairs; for each pair the sum is 10,001, so to get the answer to our problem we multiply 10,000 by 10,001 and divide this product by 2. Thus the sum of the first 10,000 counting numbers is 50,005,000. We see that we have replaced 9,999 additions by an addition followed by a multiplication and then division by two.

One way to make Gauss's arithmetic problem less abstract is to replace it by a geometry problem. Let us make associations as shown in figure 3.

Such representations of the counting numbers can be placed adjacent to each other so that they look like the bar graph formed by the solid lines in figure 4. A second "bar graph" of the same size is then made by inverting the first and placing it as shown by the dotted-line portion of figure 4.

The resulting rectangle now represents twice the sum of $1 + 2 + 3 + 4 + 5 + 6$. The area of this rectangle is 42, and this is twice the sum $1 + 2 + 3 + 4 + 5 + 6$. Notice that

$$1 + 2 + 3 + 4 + 5 + 6 = \frac{6 \times 7}{2} = 21.$$

In figure 4 we have shown how pairing the numbers corresponds to constructing a rectangular region, the area of which is twice the desired sum. Again we note that it's necessary only to visualize the rectangular region, not to construct it, since the necessary information—the length and width—is readily available. The reader might want to try the "think" method to find the sum of the first one thousand counting numbers.

It is possible, and profitable, to relate this problem to the ordinary addition table. Such a table is frequently used to note that addition is commutative by observing that the table is symmetric about a diagonal drawn from the upper left corner to the lower right corner.

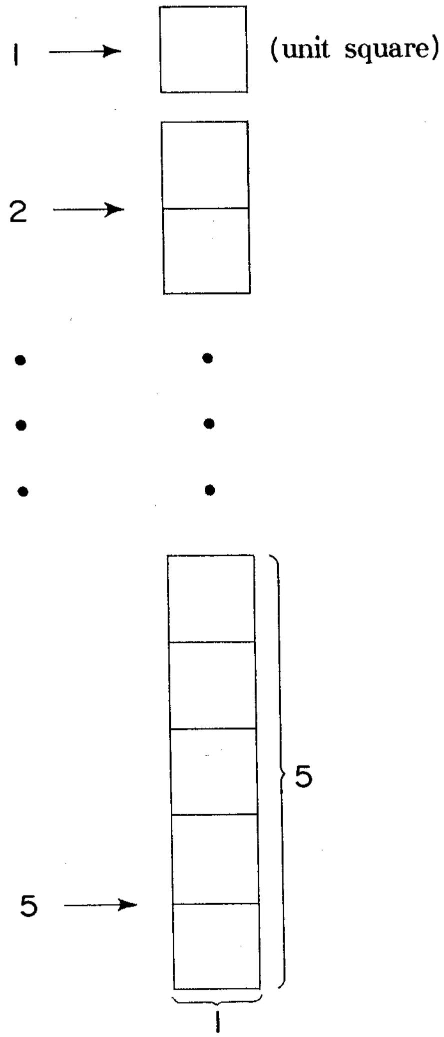

Fig. 3

Diagonally across the array of squares there are chains of squares, each with the same entry. In figure 5, we have shaded such a chain. Each block in this chain corresponds to those pairs of counting numbers whose sum is three.

If a child is to construct such a table, it will be necessary to specify the largest number that is to appear in the table. In figure 5, this specified largest number is 5.

The natural question to ask when laying out such a table is how many squares

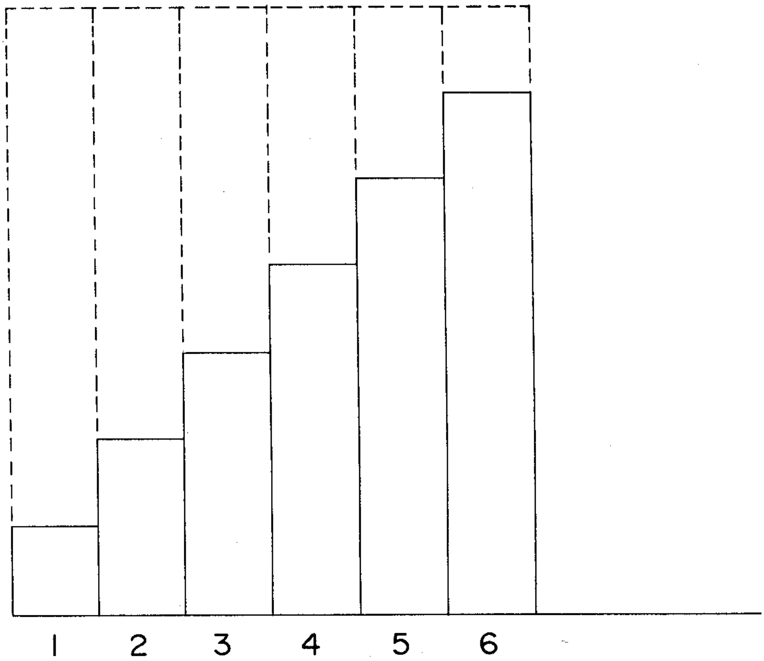

Fig. 4

are required. In figure 5, as we count the filled-in squares beginning with the bottom row, we see that the total can be represented by $1 + 2 + 3 + 4 + 5 + 6$. This is exactly the problem that Gauss solved so readily. If we make two geometric forms like the one shown by the heavy lines in figure 5 and fit them together, we get the same 6×7 rectangle as in our previous solution. The sum of the first five counting

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	
2	2	3	4	5		
3	3	4	5			
4	4	5				
5	5					

Fig. 5

numbers is 21.

An alert teacher would now ask about such sums as follows:

a) $1 + 2 + 3 + 4 + 5 + 6 + 7$ $\left(\frac{7 \times 8}{2}\right)$

b) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$ $\left(\frac{9 \times 10}{2}\right)$

c) The sum of the first one hundred counting numbers

d) The sum of the first N counting numbers $\left(\frac{N(N + 1)}{2}\right)$

The last exercise is rather difficult but worth thinking about for the students who find symbolization rather easy and challenging.

For the resourceful teacher, it is often easy to relate numbers to various geometric figures. This technique has the distinct advantages of (1) showing that mathematical problems can be solved in different ways, (2) relating arithmetic and geometry, and (3) motivating children to think about alternate ways to solve problems. Since geometry is receiving more emphasis in the elementary school, this approach to problem solving should be given serious consideration by classroom teachers.

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