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SPACES OF GENERALIZED ANALYTIC FUNCTIONS

By

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This author in some previous papers [5], [6], [7] has investigated certain properties of complex functions analytic in a strip in the complex plane. The strip was considered because it is the natural domain for functions that are Laplace transforms. MACKEY [3], ELLIOT [2], ARENS and SINGER [1] among others have given definitions of analytic functions defined on some subset of $\bar{G}\times\hat{G}$, the generalized character group of a locally compact Abelian Group. Mackey and Elliot used derivatives with respect to semigroups, Arens and Singer the duality contained in a Poisson representation. None of the above papers contain any elaboration of the properties of the class of analytic functions like the Cauchy–Riemann conditions or preservation of analyticity under uniform convergence. One result of this paper is the existence of necessary and sufficient conditions for analyticity which are analogous to the Cauchy–Riemann equations. In this paper I will also obtain those extensions which allow us to construct a Hilbert space of such functions.

NOTATION. Let G be a locally compact group (not necessarily Abelian) and $\hat{G}$ its character group. That is, $\hat{G}$ consists of those complex-valued functions, continuous on G with modulus identically 1 and with a multiplicative property. $\hat{G}$ is also a locally compact group and the topology is that of uniform convergence on compact subsets of G. $\bar{G}$ denotes the set of real, continuous linear functionals on G. A subset K, of $\bar{G}$ is said to be large convex if it is convex, contains the zero element and the closed linear span is $\bar{G}$. $\bar{G}\times\bar{G}$ becomes a complex vector space by defining $(u+iv)(x_1, x_2) = (ux_1 - vx_2, vx_1 + ux_2)$ for $u+iv$ complex and $(x_1, x_2)\in\bar{G}\times\bar{G}$. Finally for each $x\in\bar{G}$, define a one-parameter subgroup of $\hat{G}$ by $x[u] = \exp(iux)$, u real.

DEFINITION (MACKEY [4]). Let K be a large convex subset of $\bar{G}$ and x_0 an interior point of K (i.e. there exists $r>1$ such that $rx\in K$). Let $F(x, y)$ be a complex-valued function defined on $K\times\hat{G}$. It is said to be analytic at (x_0, y_0) if

$$\text{(i)} \qquad \operatorname*{Lim}_{u\to 0} \frac{F(x_0+ux_1, y_0x_2[u]) - F(x_0, y_0)}{u} = F_{x_1, x_2}(x_0, y_0)$$

exists for every $(x_1, x_2)\in\bar{G}$.

(ii) The above limit is a complex-homogeneous function of (x_1, x_2). As will be shown the complex-homogeneity condition is analogous to requiring that the partial derivatives satisfy Cauchy–Riemann like conditions.

LEMMA 1.1. *If $F(x, y)$ is analytic at $(x, y) \in \bar{G} \times \hat{G}$ and $U(x, y)$, $V(x, y)$ denote real valued functions such that $F(x, y) = U(x, y) + iV(x, y)$ then $U_{(x_1, x_2)}(x, y)$, $V_{(x_1, x_2)}(x, y)$ exist as complex-homogeneous functions of (x_1, x_2) and*

$$U_{(x_1, 0)} = V_{(0, x_1)}, \qquad U_{(0, x_1)} = -V_{(x_1, 0)}$$

for all $x_1 \in \bar{G}$.

PROOF. Since $0 \in \bar{G}$, $F_{(x_1, 0)}$ exists and $F_{(x_1, 0)} = U_{(x_1, 0)} + iV_{(x_1, 0)}$. Since it follows easily from the definition that the differentiation is linear. Further $i(x_1, 0) = (0, x_1)$ so that

$$iF_{(x_1, 0)} = iU_{(x_1, 0)} - V_{(x_1, 0)} = F_{(0, x_1)} = U_{(0, x_1)} + iV_{(0, x_1)}$$

and hence

$$U_{(x_1, 0)} = -V_{(x_1, 0)}, \qquad V_{(0, x_1)} = U_{(x_1, 0)}.$$

LEMMA 1.2. *Let $(x, y) \in \bar{G} \times \hat{G}$, N a convex neighbourhood of (x, y) such that at each point (x', y') in N and for all $x_1 \in \bar{G}$, $U_{(x_1, 0)}(x', y')$ exists. Then there is a $\delta > 0$ such that for $0 < \alpha < \delta$, there is a $0 < \beta(\alpha) < \alpha$ where*

$$U(x + \alpha x_1, y) - U(x, y) = \alpha U_{(x_1, 0)}(x + \beta x_1, y).$$

PROOF. Let $g(w) = U(x + wx_1, y)$, then there exists $\delta > 0$ such that $g'(w)$ exists for $0 \leqq w \leqq \delta$ since

$$g'(w) = \lim_{\Delta w \to 0} \frac{U(x + (w + \Delta w)x_1, y) - U(x + wx_1, y)}{\Delta w} =$$

$$= \lim_{\Delta w \to 0} \frac{U(x + wx_1 + \Delta w x_1, y) - U(x + wx_1, y)}{\Delta w} = U_{(x_1, 0)}(x + wx_1, y)$$

which by hypothesis exists for all $(x + wx_1, y) \in N$. δ may be taken to be any positive number such that $(x + \delta x_1, y) \in N$, by convexity $(x + \alpha x_1, y) \in N$ for $0 \leqq \alpha \leqq \delta$. Applying the Mean-Value Theorem to g we have

$$g(\alpha) - g(0) = \alpha g'(\beta)$$

for some $0 < \beta < \alpha$, or

$$U(x + \alpha x_1, y) - U(x, y) = \alpha U_{(x_1, 0)}(x + \beta x_1, y).$$

COROLLARY. *If further $U_{(x_1, 0)}$ is a complex homogeneous function of $(x_1, 0)$ then*

$$U(x, y) - U(x + \alpha x_1, y) = i\alpha U_{(0, x_1)}(x + \beta x_1, y).$$

LEMMA 1.3. *Let (x, y) be in $\bar{G} \times \hat{G}$ and N a convex neighbourhood of (x, y) such that for all (x', y') in N and all x_1 in $\bar{G}$, $U_{(0, x_1)}(x', y')$ exists. Then there exists $\delta > 0$ such that for all $0 < \alpha < \delta$, there exists $0 < \beta(\alpha) < \alpha$ such that*

$$U(x, yx_1[\alpha]) - U(x, y) = \alpha U_{(0, x_1)}(x, yx_1[\beta]).$$

PROOF. Let $h(w) = U(x, yx_1[w])$ and the proof proceeds as for Lemma 1 if we note that

$$U(x, yx_1[w + \Delta w]) = U(x, yx_1[w]x_1[\Delta w]).$$

COROLLARY. *If further $U_{(0, x_1)}$ is a complex homogeneous function of $(0, x_1)$ then*

$$U(x, y) - U(x, yx_1[\alpha]) = -i\alpha U_{(x_1, 0)}(x, yx_1[\beta]).$$

COROLLARY. *If further $U_{(0, x_1)}$ is a complex homogeneous function of $(0, x_1)$, (and hence $U_{(x_1, 0)}$ is of $(x_1, 0)$) then*

$$U(x, yx_1[\alpha]) - U(x + \alpha x_1, y) = i\alpha\ [U_{(x_1, 0)}(x, yx_1[\beta']) - U_{(0, x_1)}(x + \beta'' x_1, y)]$$

for some $0 < \beta' < \alpha$, $0 < \beta'' < \alpha$ or

$$U(x + \alpha x_1, y) - U(x, yx_1[\alpha]) = \alpha\ [U_{(x_1, 0)}(x + \beta'' x_1, y) - U_{(0, x_1)}(x, yx_1[\beta'])].$$

LEMMA 1.4. *Let $F(x, y) = U(x, y) + iV(x, y)$ and suppose that for all $x_1 \in \bar{G}$, $U(x', y')$, $V(x', y')$, $U_{(x_1, 0)}(x', y')$, $V_{(x_1, 0)}(x', y')$ exist and are continuous in a convex neighbourhood of (x, y). Then if $U_{(0, x_1)}(x, y) = -V_{(x_1, 0)}(x, y)$, $U_{(x_1, 0)}(x, y) = V_{(0, x_1)}(x, y)$ for all x_1 in $\bar{G}$, $F(x, y)$ is analytic at (x, y).*

PROOF. It is clearly sufficient to show that

$$(\alpha + i\beta) \lim_{u \to 0} \frac{F(x + ux_1, yx_2[u]) - F(x, y)}{u}$$

and

$$\lim_{u \to 0} \frac{F(x + \alpha ux_1 - \beta ux_2, yx_2[\alpha u]x_1[\beta u]) - F(x, y)}{u}$$

exist and are equal for all $\alpha + i\beta \in C$ and $(x_1, x_2) \in \bar{G} \times \bar{G}$. Writing $F(x, y) = U(x, y) + iV(x, y)$ we may consider U and V separately.

(1) Consider

$$\frac{U(x + \alpha ux_1 - \beta ux_2, yx_2[\alpha u]x_1[\beta u])) - U(x, y)}{u} =$$

$$= \alpha \left[\frac{U(x + \alpha ux_1 - \beta ux_2, yx_2[\alpha u]x_1[\beta u]) - U(x - \beta ux_2, yx_2[\alpha u]x_1[\beta u])}{\alpha u} \right] +$$

$$+ \alpha \frac{U(x - \beta ux_2, yx_2[\alpha u]x_1[\beta u]) - U(x - \beta ux_2, yx_1[\beta u])}{\alpha u} +$$

$$+ \beta \left[\frac{U(x - \beta ux_2, yx_1[\beta u]) - U(x - \beta ux_2, y)}{\alpha u} \right] - \beta \left[\frac{U(x - \beta ux_2, y) - U(x, y)}{-\beta u} \right].$$

If we apply Lemma 2 and Lemma 3 then we obtain

$$\frac{U(x + \alpha u x_1 - \beta x_2, yx_2[\alpha u]x_1[\beta u]) - U(x, y)}{u} =$$

$$= \alpha U_{(x_1, 0)}(x - \beta u x_2 + \delta x_1, yx_2[\alpha u]x_1[\beta u]) + \alpha U_{(0, x_2)}(x - \beta u x_2, yx_2[\beta u]x_2(\delta')) +$$
$$+ \beta U_{(0, x_1)}(x - \beta u x_2, yx_1[\delta'']) - \beta U_{(x_2, 0)}(x - \delta''' x_2, y)$$

where $0 < \delta, \delta' < \alpha u$, $0 < \delta'', \delta''' < \beta u$. Using the continuity of the partials however, as $u \to 0$ we obtain

$$\alpha U_{(x_1, 0)}(x, y) + \alpha U_{(0, x_2)}(x, y) + \beta U_{(0, x_1)}(x, y) - \beta U_{(x_2, 0)}(x, y).$$

(2) Proceeding in a similar fashion for

$$\frac{V(x + \alpha u x_1 - \beta u x_2, yx_2[\alpha u]x_1[\beta u]) - V(x, y)}{u}$$

we obtain, as $u \to 0$

$$\alpha V_{(x_1, 0)}(x, y) + \alpha V_{(0, x_2)}(x, y) + \beta V_{(0, x_1)}(x, y) - \beta V_{(x_2, 0)}(x, y).$$

Now utilizing the "Cauchy–Riemann" conditions we obtain

$$\lim_{u \to 0} \frac{F(x + \alpha u x_1 - \beta u x_2, yx_2[\alpha u]x_1[\beta u]) - F)x, y)}{u} =$$
$$= \alpha U_{(x_1, 0)}(x, y) + \alpha U_{(0, x_2)}(x, y) - \beta V_{(x_1, 0)}(x, y) - \beta V_{(0, x_2)} +$$
$$+ \alpha V_{(x_1, 0)}(x, y) + \alpha V_{(0, x_2)}(x, y) + i\beta U_{(x_1, 0)}(x, y) + i\beta U_{(0, x_2)}(x, y) =$$
$$= (\alpha + i\beta)[U_{(x_1, 0)}(x, y) + U_{(0, x_2)}(x, y)] + (\alpha + i\beta)[iV_{(x_1, 0)}(x, y) + iV_{(0, x_2)}] =$$
$$= (\alpha + i\beta)[U_{(x_1, x_2)}(x, y) + iV_{(x_1, x_2)}(x, y)].$$

Since

$$U_{(x_1, 0)}(x, y) + U_{(0, x_2)}(x, y) = U_{(x_1, x_2)}(x, y)$$

and

$$V_{(x_1, 0)}(x, y) + V_{(0, x_2)}(x, y) = V_{(x_1, x_2)}(x, y).$$

The proof is now complete. We note that another way of writing the Cauchy–Riemann equations is

$$F_{(x_1, 0)}(x, y) + iF_{(0, x_1)} = 0.$$

DEFINITION 1. Let K be a large convex subset of $\bar{G}$. Define $H^2(K)$ to be

$$\{F \mid F \text{ analytic at each point of } K \times \hat{G}, \sup_{x \in K} \int |F(x, y)|^2 dy < \infty\}$$

(the integral refers to the left-invariant Haar measure on G).

THEOREM 2. *$H^2(K)$ is a Hilbert space with point-wise addition and inner product*

$$(F_1, F_2) = \sup_{x \in K} \int_{\hat{G}} F_1(x, y)\overline{F_2(x, y)}dy \qquad (\|F\| = \sqrt{(F, F)})$$

The proof will follow from a series of Lemmas.

LEMMA 2.1. *If $\{F_n\}$ is a sequence in $H^2(K)$ such that $||F_n - F_m|| \to 0$ as $n, m \to \infty$ then $F_n - F_m$ converges uniformly to zero on compact subsets of $K \times \hat{G}$.*

PROOF. Suppose $|| F_n - F_m || \to 0$ and there exists a compact K such that $F_n - F_m \to 0$ uniformly on K. Since $|| F_n - F_m || \to 0$, $| F_n(x, y) - F_m(x, y) |$ is bounded on K for each n, m. Let

$$g_{nm}(K) = \sup_{(x, y) \in K} |F_n(x, y) - F_m(x, y)|.$$

Since K is compact, for each n, m there exists $p_{nm} = (x_{nm}, y_{nm})$ in K such that

$$g_{nm}(K) = | F_n(p_{nm}) - F_m(p_{nm}) | .$$

Now since $F_n - F_m \not\to 0$ uniformly on K, there exist two unbounded sequences $\{n_k\}$, $\{m_k\}$ such that $g_{n_k m_k}(K) > \varepsilon$ for all k. Let K_ε be the closure of $\{p_k\}$, $p_{n_k m_k} = p_k$. By the continuity of $| F_n(x, y) - F_m(x, y) |$ for each k there is a neighbourhood N_k of p_k such that $p \in N$ implies $| f_{n_k}(p) - f_{m_k}(p) | > \varepsilon/2$.

K_ε is covered by the collection of such neighbourhoods and as a closed subset of a compact set, K_ε is compact hence there exists a finite subscover $N_{k_1}, \dots, N_{k_J}$. Let

$$M_{k_i} = \{y \mid y \in \hat{G}, \exists\, x \ni (x, y) \in N_{k_i}\} .$$

Now we note that

$$|| F_n - F_m || \geqq \int_{\hat{G}} |F_n(x, y) - F_m(x_k, y) |^2 dy$$

for all x_k, and hence

$$|| F_n - F_m || \geqq \int_{M_{k_i}} | F_n(x_{k_i}, y) - F_m(x_{k_i}, y) |^2 dy \geqq \frac{\varepsilon}{2} \mu(M_{k_i})$$

for $n = n_{k_i}$, $m = m_{k_i}$. If now $\delta = \min (\mu(M_{k_i}), \dots, \mu(M_{k_j}))$ then

$$|| F_n - F_m || \geqq \varepsilon\delta/2 \qquad \text{for} \qquad n = n_k, \qquad m = m_k \qquad \text{all} \qquad k,$$

which contradicts $|| F_n - F_m || \to 0$ and we conclude that $F_n - F_m \to 0$ uniformly on K.

LEMMA 2.2. *Let F be analytic at each point of $K \times \hat{G}$, K a large convex subset of $\bar{G}$. If K is a compact subset of $K \times \hat{G}$ then F is uniformly analytic on K, i.e. given $\varepsilon > 0$, $(x_1, x_2) \in \bar{G} \times \bar{G}$ there exists $\delta > 0$ such that*

$$\frac{| F(x + ux_1, yx_2[u]) - F(x, y)}{u} - F_{(x_1, x_2)}(x, y) < \varepsilon$$

for $0 < | u | < \delta$ and all $(x, y) \in K$.

PROOF. Let $\varepsilon > 0$, $(x_1, x_2) \in \bar{G} \times \bar{G}$, then for each $(x_0, y_0) \in K$ set

$$N(x_0, y_0) = \left\{(x, y) \mid \left| \frac{F(x + ux_1, yx_2[u]) - F(x, y)}{u} - F_{(x_1 x_2)}(x, y) \right| < \varepsilon \text{ if } 0 < |u| < \delta \right\}$$

(δ is taken to be any non-zero value such that $0 < |u| < \delta$ implies

$$\left| \frac{F(x_0 + ux_1, y_0 x_2[u]) - F(x_0, y_0)}{u} - F_{(x_1, x_2)}(x_0, y_0) \right| < \varepsilon).$$

Then $\bigcup_{(x_0, y_0) \in K} N(x_0, y_0)$ is an open covering of K, which is compact. Then there exist a finite number of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ associated δ's $\delta_1, \dots, \delta_n$ and neighbourhoods $N(x_1, y_1), \dots, N(x_n, y_n)$ which is a finite subcovering of K. Let $\delta = \min(\delta_1, \dots, \delta_n)$ and the result follows.

LEMMA 2.3. *Let $F(x, y)$ be defined in a neighbourhood of $(x, y) \in \bar{G} \times \hat{G}$. Then a necessary and sufficient condition that F be analytic (in the sense of Definition 1) is that $g(z)$ be analytic at $z = 0$ for all $(x_1, x_2) \in \bar{G} \times \bar{G}$ where $g_{(x_1, x_2)}(z) = F(x + ux_1 - vx_2, yx_2[u]x_1[v])$, $z = u + iv$.*

PROOF. (a) Writing $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$ as is customary then $g_{x_1}(z)$ analytic at $z = 0$ implies

$$\left.\frac{\partial g}{\partial z}\right|_{z=0} = 0.$$

However

$$\left.\frac{\partial g}{\partial u}\right|_{z=0} = \lim_{u \to 0} \frac{F(x + ux_1, yx_2[u]) - F(x, y)}{u} = F_{(x_1, x_2)}(x, y)$$

and

$$\left.\frac{\partial g}{\partial v}\right|_{z=0} = \lim_{v \to 0} \frac{F(x - vx_2, yx_1[v]) - F(x, y)}{v} = F_{(-x_2, x_1)}(x, y).$$

Hence $\left.\frac{\partial g}{\partial z}\right|_{z=0} = 0$ implies

$$F_{(x_1, x_2)}(x, y) + iF_{(-x_2, x_1)}(x, y) = 0$$

or

$$F_{(x_1, x_2)}(x, y) = -iF_{(-x_2, x_1)}(x, y)$$

which together with the observations that $-i(-x_2, x_1) = (x_1, x_2)$ provides the complex homogeneity required. The existence of the derivative follows from the existence of the derivative of g. Since the preceding assertions are reversible the fooɹd of the theorem is complete.

LEMMA 2.4. *Let $\{F_n(x, y)\}$ be a sequence of complex-valued functions analytic at each point of $K \times \hat{G}$, K large convex in $\bar{G}$. If $\{F_n\}$ converges uniformly on compact subsets of $K \times \hat{G}$ then the limit function is analytic at each point of $K \times \hat{G}$.*

PROOF. Let $(x_0, y_0) \in K \times \hat{G}$, $(x_1, x_2) \in \bar{G} \times \bar{G}$ and $\delta > 0$.

$$M_{(x_1, x_2)}(\delta) = \{(x, y) | x = x_0 + ux_1 - vx_2, y = y_0 x_2[u] x_1[v], (x, y) \in M, \\ |z| \leqq \delta, (x, y) \in K \times \hat{G}\},$$

then $M_{(x_1, x_2)}(\delta)$ is compact and is nonempty for some $\delta \neq 0$. We see then that $\{F_n\}$ converging uniformly on $M_{(x_1, x_2)}(\delta)$ is equivalent, to $\{g_n\}$ converging uniformly on $|z| \leqq \delta$ where

$$g_n(z) = F_n(x_0 + ux_1 - vx_2, y_0 x_2[u] x_1[v]).$$

Now $g_n(z)$ is analytic for $|z| \leqq \delta$ if and only if F_n is analytic in $M_{(x_1, x_2)}(\delta)$ with respect to (x_1, x_2) by Lemma 2.3. Therefore $\{g_n\}$ has an analytic limit for $|z| \leqq \delta$ if and only if $\{F_n\}$ has an analytic limit at (x_0, y_0). Since (x_0, y_0) and (x_1, x_2) were arbitrary, the proof is complete.

PROOF OF THEOREM 2. By Lemma 2.1 a Cauchy sequence in $H^2(K)$ is a Cauchy sequence in the topology of uniform convergence on compact subsets of $K \times \hat{G}$. From Lemma 2.4 the limit under this latter topology is analytic. From the usual L^2 topology we have that the limit function must be in $H^2(K)$. The algebraic closure of $H^2(K)$ follows in the usual way.

THEOREM 3. *If $F(x, y)$ is analytic at $(x_0, y_0) \in K \times \hat{G}$, for all n, $g(z_1, z_2, \dots, z_n)$ is analytic at $(z_1, z_2, \dots, z_n) = (0, \dots, 0)$ where*

$$g(z_1, \dots, z_n) = F(x_0 + \sum_{i=1}^{n} (u_i x_i - v_i x_{i+n}), y_0 \prod_{i=1}^{n} x_i[v_i] x_{i+n}[u_i]), \\ (x_i, x_{i+n}) \in \bar{G} \times \bar{G}, \qquad i = 1, \dots n.$$

PROOF. By n applications of Lemma 2.3, g is analytic in z_i for each i and hence by the Osgood–Hartog Theorem [3] g is analytic.

COROLLARY 3.1. *If F is analytic at (x_0, y_0) then for all $(x_1, x_2) \in \bar{G} \times \bar{G}$, $F_{(x_1, x_2)}$ is analytic at (x_0, y_0).*

PROOF. Let $n = 2$ in Theorem 3 then $g(z_1, z_2)$ is analytic at $(0, 0)$ and hence the double and iterated limits on the derivatives exist and are equal, i.e.

$$\left.\frac{\partial^2 g}{\partial \bar{z}_1 \partial \bar{z}_2}\right|_{(0,0)} = \left.\frac{\partial^2 g}{\partial \bar{z}_2 \partial \bar{z}_1}\right|_{(0,0)} = 0.$$

Evaluating these as in the proof of Lemma 2.3 we have

$$(F_{(x_1, x_3)})_{(x_2, x_4)} = (F_{(x_2, x_4)})_{(x_1, x_3)}$$

and both exist and are complex homogeneous functions of (x_1, x_3), (x_2, x_4). By repeated applications of this corollary we have that F analytic implies the existence of all higher orders of derivatives.

If we re-define analyticity then we can obtain a weak form of Osgood—Hartog Theorem for these functions.

DEFINITION 4. Let $F(x, y)$ be defined in a neighbourhood of $(x_0, y_0) \in \bar{G} \times \hat{G}$. Then F is said to be analytic in x_1 at (x_0, y_0) if

$$\lim_{u \to 0} \frac{F(x_0 + ux_1, yx_1[u]) - F(x, y)}{u} = F_{(x_1, x_1)}(x_0, y_0)$$

exists and is a complex homogeneous function of (x_1, x_1).

THEOREM 5. (Weak Osgood—Hartog.) *If F is "analytic in x_1 at (x_0, y_0)" for all $x_1 \in \bar{G}$ then F is analytic (in the sense of Definition 1).*

PROOF. We note that

$$(x_1, x_2) = \frac{1-i}{2}(x_1, x_1) + \frac{1+i}{2}(x_2, x_2)$$

hence

$$F_{(x_1, x_2)} = \frac{1-i}{2} F_{(x_1, x_1)} + \frac{1+i}{2} F_{(x_2, x_2)}$$

and $F_{(x_1, x_2)}$ is a complex homogeneous function of (x_1, x_2) since $F_{(x_1, x_1)}$ and $F_{(x_2, x)_2}$ are complex homogeneous functions of (x_1, x_1), (x_2, x_2) respectively.

In conclusion we note two other results. Lemma 2.3 asserts that Definition 1 and Definition 4.10 [2] are equivalent. $H^2(K)$ is a subset of the set of L^2-analytic functions as defined by MACKEY [4] and hence by the Theorem, pp. 160 [4] each is the generalized Laplace Transform of a function in L^2 up to an equivalence relation.

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