The OLS Estimator Formula for the Parameters of a Multiple Regression Model

Consider the OLS estimator for b_1. The normal equation corresponding to variable X_1 is given by

$$\sum_t Y_t X_{1t} = \hat{b}_0 \sum_t X_{1t} + \hat{b}_1 \sum_t X_{1t}^2 + \hat{b}_2 \sum_t X_{2t} X_{1t} + ... + \hat{b}_K \sum_t X_{Kt} X_{1t}$$

Since $X_{1t} = \hat{X}_{1t} + \hat{V}_{1t}$, we can replace X_{1t} by $\hat{X}_{1t} + \hat{V}_{1t}$:

$$\Rightarrow \sum_t Y_t \left(\hat{X}_{1t} + \hat{V}_{1t}\right) = \hat{b}_0 \sum_t \left(\hat{X}_{1t} + \hat{V}_{1t}\right) + \hat{b}_1 \sum_t \left(\hat{X}_{1t} + \hat{V}_{1t}\right)^2$$

$$+\hat{b}_2 \sum_t X_{2t} \left(\hat{X}_{1t} + \hat{V}_{1t}\right) + ... + \hat{b}_K \sum_t X_{Kt} \left(\hat{X}_{1t} + \hat{V}_{1t}\right)$$

$$\Rightarrow \sum_t Y_t \hat{X}_{1t} + \sum_t Y_t \hat{V}_{1t} = \hat{b}_0 \sum_t \hat{X}_{1t} + \hat{b}_1 \sum_t \hat{X}_{1t}^2 + \hat{b}_2 \sum_t X_{2t} \hat{X}_{1t}$$

$$+ ... + \hat{b}_K \sum_t X_{Kt} \hat{X}_{1t} + \hat{b}_1 \sum_t \hat{V}_{1t}^2,$$

since $\sum_t \hat{V}_{1t} = 0$, and $\sum_t X_{it} \hat{V}_{1t} = 0 \;\text{ for } i \neq 1$. Now

$$\sum_t Y_t \hat{X}_{1t} = \hat{b}_0 \sum_t \hat{X}_{1t} + \hat{b}_1 \sum_t \hat{X}_{1t}^2 + \hat{b}_2 \sum_t X_{2t} \hat{X}_{1t} + \hat{b}_K \sum_t X_{Kt} \hat{X}_{1t}$$

because $Y_t = \hat{b}_0 + \hat{b}_1 X_{1t} + ... + \hat{b}_K X_{Kt} + \hat{u}_t$ and $\sum_t \hat{u}_t \hat{X}_{1t} = 0$. Therefore, $\sum_t Y_t \hat{V}_{1t} = \hat{b}_1 \sum_t \hat{V}_{1t}^2$ so that

$$\boxed{\hat{b}_1 = \frac{\sum_t Y_t \hat{V}_{1t}}{\sum_t \hat{V}_{1t}^2}}$$

Thus $\hat{b}_1$ is the estimated coefficient from a simple regression of Y_t on $\hat{V}_{1t}$ (the purged component of X_{1t} that is uncorrelated with the other variables). In general for the *ith* coefficient estimator we have

$$\boxed{\hat{b}_i = \frac{\sum_t Y_t \hat{V}_{it}}{\sum_t \hat{V}_{it}^2}} \qquad i = 1, ..., K$$

$$\boxed{\hat{b}_0 = \bar{Y} - \sum_{i=1}^{K} \hat{b}_i \bar{X}_i}$$