Econ 522a

Conditional Forecasting Variances

$$
\begin{aligned}
Var\left(e_f\right) &= Var\left(X_f\beta - \hat{X}_f\hat{\beta}\right) + Var\left(u_f\right) + 2\underbrace{Cov\left[\left(X_f\beta - \hat{X}_f\hat{\beta}\right), u_f\right]}_{0} \\
&= Var\left(X_f\beta - \hat{X}_f\hat{\beta}\right) + \sigma_u^2 \qquad (1)
\end{aligned}
$$

$$
X_f\beta - \hat{X}_f\hat{\beta} = \left(X_f - \hat{X}_f\right)\beta + X_f\left(\beta - \hat{\beta}\right) + \left(\hat{X}_f - X_f\right)\left(\beta - \hat{\beta}\right) \qquad (2)
$$

therefore,

$$
\begin{aligned}
Var\left(X_f\beta - \hat{X}_f\hat{\beta}\right) &= \underbrace{Var\left[\left(X_f - \hat{X}_f\right)\beta\right]}_{(3a)} + \underbrace{Var\left[X_f\left(\beta - \hat{\beta}\right)\right]}_{(3b)} \qquad (3) \\
&\quad + \underbrace{Var\left[\left(\hat{X}_f - X_f\right)\left(\beta - \hat{\beta}\right)\right]}_{(3c)} \\
&\quad + 2\underbrace{Cov\left[\left(X_f - \hat{X}_f\right)\beta, X_f\left(\beta - \hat{\beta}\right)\right]}_{(3d)} \\
&\quad + 2\underbrace{Cov\left[\left(X_f - \hat{X}_f\right)\beta, \left(\hat{X}_f - X_f\right)\left(\beta - \hat{\beta}\right)\right]}_{(3e)} \\
&\quad + 2\underbrace{Cov\left[X_f\left(\beta - \hat{\beta}\right), \left(\hat{X}_f - X_f\right)\left(\beta - \hat{\beta}\right)\right]}_{(3f)}
\end{aligned}
$$

$$
\begin{aligned}
Var\left[\left(X_f-\hat{X}_f\right)\beta\right] &= E\left[\beta'\left(X_f-\hat{X}_f\right)'\left(X_f-\hat{X}_f\right)\beta\right] \\
&= \beta' E\left[\left(X_f-\hat{X}_f\right)'\left(X_f-\hat{X}_f\right)\right]\beta \\
&= \beta'\Sigma_{\hat{X}_f-X_f}\beta \qquad \text{(3a)}
\end{aligned}
$$

Conditioning on X_f yields

$$
\begin{aligned}
Var\left[X_f\left(\beta-\hat{\beta}\right)\right] &= E\left[X_f\left(\beta-\hat{\beta}\right)\left(\beta-\hat{\beta}\right)' X_f'\right] \\
&= X_f E\left[\left(\beta-\hat{\beta}\right)\left(\beta-\hat{\beta}\right)'\right] X_f' \\
&= X_f \Sigma_{\hat{\beta}} X_f' \qquad \text{(3b)}
\end{aligned}
$$

where $\Sigma_{\hat{\beta}} = Var(\hat{\beta}) = \sigma_u^2\left(X'X\right)^{-1}$.

$$
\begin{aligned}
Var\left[\left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right] &= E\left\{\left[\left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right]\left[\left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right]'\right\} \\
&= E\ \text{tr}\left[\underbrace{\left(\hat{X}_f-X_f\right)}_{A}\underbrace{\left(\beta-\hat{\beta}\right)\left(\beta-\hat{\beta}\right)'\left(\hat{X}_f-X_f\right)'}_{B}\right] \\
&= E\ \text{tr}\left[\underbrace{\left(\beta-\hat{\beta}\right)\left(\beta-\hat{\beta}\right)'\left(\hat{X}_f-X_f\right)'}_{B}\underbrace{\left(\hat{X}_f-X_f\right)}_{A}\right] \\
&= \text{tr}\left\{E\left[\left(\beta-\hat{\beta}\right)\left(\beta-\hat{\beta}\right)'\right]\cdot E\left[\left(\hat{X}_f-X_f\right)'\left(\hat{X}_f-X_f\right)\right]\right\} \\
&= \text{tr}\left[\Sigma_{\hat{\beta}}\cdot\Sigma_{\hat{X}_f-X_f}\right] \qquad \text{(3c)}
\end{aligned}
$$

$$
\begin{aligned}
Cov\left[\left(X_f-\hat{X}_f\right)\beta, X_f\left(\beta-\hat{\beta}\right)\right] &= E\left[\left(X_f-\hat{X}_f\right)\beta X_f\left(\beta-\hat{\beta}\right)\right] \\
&= E\left[\left(X_f-\hat{X}_f\right)\beta X_f\right]\cdot E\left(\beta-\hat{\beta}\right) \\
&= 0 \qquad \text{(3d)}
\end{aligned}
$$

$$
\begin{aligned}
Cov\left[\left(X_f-\hat{X}_f\right)\beta, \left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right] &= E\left[\left(X_f-\hat{X}_f\right)\beta\left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right] \\
&= E\left[\left(X_f-\hat{X}_f\right)\beta\left(\hat{X}_f-X_f\right)\right]\cdot E\left(\beta-\hat{\beta}\right) \\
&= 0 \qquad \text{(3e)}
\end{aligned}
$$

Conditioning on $X_{f\ ,}$ yields

$$
\begin{aligned}
Cov\left[X_f\left(\beta-\hat{\beta}\right), \left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right] &= E\left[X_f\left(\beta-\hat{\beta}\right)\left(\hat{X}_f-X_f\right)\left(\beta-\hat{\beta}\right)\right] \\
&= E\ \text{tr}\left[\underbrace{X_f\left(\beta-\hat{\beta}\right)\left(\hat{X}_f-X_f\right)}_{A}\underbrace{\left(\beta-\hat{\beta}\right)}_{B}\right] \\
&= E\ \text{tr}\left[\underbrace{\left(\beta-\hat{\beta}\right)}_{B}\underbrace{X_f\left(\beta-\hat{\beta}\right)\left(\hat{X}_f-X_f\right)}_{A}\right] \\
&= \text{tr}\ \left\{E\left[\left(\beta-\hat{\beta}\right)X_f\left(\beta-\hat{\beta}\right)\right]\cdot E\left(\hat{X}_f-X_f\right)\right\} \\
&= 0 \qquad \text{(3f)}
\end{aligned}
$$

After making appropriate substitutions in (3) and (1), one obtains

$$
\begin{aligned}
Var\left(e_f\right) &= \beta'\Sigma_{\hat{X}_\mathsf{f}-X_\mathsf{f}}\beta + X_f\Sigma_{\hat{\beta}}X_f' + \text{ tr }\left[\Sigma_{\hat{\beta}}\cdot\Sigma_{\hat{X}_\mathsf{f}-X_\mathsf{f}}\right] + \sigma_u^2 \\
&= \beta'\Sigma_{\hat{X}_\mathsf{f}-X_\mathsf{f}}\beta + \sigma_u^2\left[X_f\left(X'X\right)^{-1}X_f'+1\right] + \text{ tr }\left[\Sigma_{\hat{\beta}}\cdot\Sigma_{\hat{X}_\mathsf{f}-X_\mathsf{f}}\right]. \qquad \text{(4)}
\end{aligned}
$$

Note that $Var\left(e_f\right) > \sigma_u^2\left[X_f\left(X'X\right)^{-1}X_f' + 1\right]$, i.e. the variance of the forecast error when X_f is unknown exceeds the variance of the forecast error when X_f is known.

In practice one would use

$$\hat{\sigma}_{e_f}^2 = \hat{\beta}'\hat{\Sigma}_{\hat{X}_f - X_f}\hat{\beta} + \hat{\sigma}_u^2\left[\hat{X}_f\left(X'X\right)^{-1}\hat{X}_f' + 1\right] + \text{ tr }\left[\hat{\Sigma}_{\hat{\beta}} \cdot \hat{\Sigma}_{\hat{X}_f - X_f}\right]$$