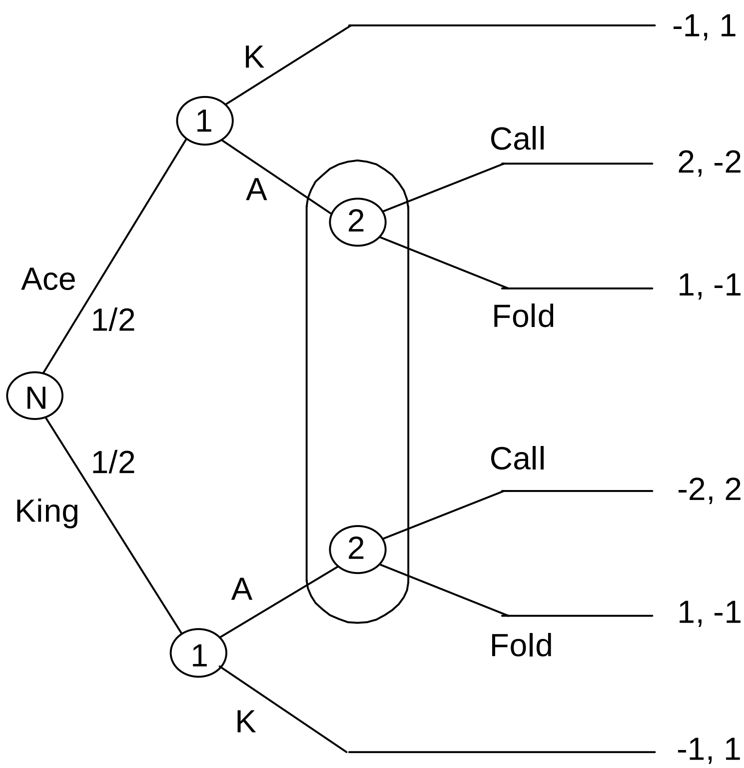
A Simple Poker Game

There are two players. Each player places $1 into the pot as his ante. Then Player 1 is dealt a single card from a deck consisting of only Aces and Kings, in equal numbers. Player 1 is allowed to look at his card, but Player 2 cannot see it. Then Player 1 can either Bet (saying "I have an Ace") or Fold (saying "I have a King"). If he folds, Player 2 wins the pot, thus winning $1 from Player 1 (i.e., he wins Player 1's ante). If instead of folding Player 1 bets, then Player 2 has a choice: she can either Call, which requires Player 2 to match Player 1's $1 bet, or Fold. If Player 2 folds, then Player 1 wins the pot, thus winning $1 from Player 2 (i.e., he wins Player 2's ante). But if Player 2 calls, then Player 1 must reveal his card; if the card is an Ace then Player 1 wins the entire pot (which has $2 of Player 2's money), and if the card is a King then Player 2 wins the entire pot (which has $2 of Player 1's money). Note that Player 1 is not required to be honest: he can bet, saying "I have an Ace," even if he has a King; and he can fold, saying "I have a King," even if he has an Ace.

Here is the extensive form representation of the game (i.e., the game tree):



Notice that we've had to include an "information set" in the game: When Player 2 gets to move, she doesn't know which node the game is at. She knows that Player 1 has *said* he has an Ace, but she doesn't know whether Player 1 was actually *dealt* an Ace, or whether he was actually dealt a King. The meaning of the information set containing both of Player 2's decision nodes is that, because Player 2 doesn't know which node she's really at, any strategy she uses cannot prescribe different actions at the two nodes. She can't, for example, choose a strategy that says "Fold when Player 1 has an Ace and Call when Player 1 has a King." She has to use a strategy that simply says "at this information set, when Player 1 *says* he has an Ace, I do this."

One of the possible strategies Player 1 could use is to always play "honest" – i.e., to always announce his card truthfully. If Player 1 were to use that strategy, then Player 2's best response would be to fold whenever she gets the chance to move (because she would be sure that Player 1 has an ace). Then whenever Player 1 has an ace (which is half the time), he will win $1 from Player 2; and whenever Player 1 has a king (also half the time), Player 2 will win $1 from Player 1. Consequently this strategy profile results in a zero expected value for each player. Thus, we know that Player 1 has a strategy that can guarantee himself an expected payoff of at least zero. And of course that means that Player 2 cannot guarantee herself a positive expected return. But perhaps Player 1 can do *better* than this: perhaps playing his "honest" strategy is not a best response to Player 2's strategy of always folding. On the other hand, if Player 1 *can't* do better, then Player 2, by always folding, guarantees herself a zero expected return, and the game is a "fair game." Can Player 1 do better, or not?

Let's look at the strategic form of the game. For that, we need to identify all the possible pure strategies available to each player. Recall that a pure strategy is a prescription that says which of your actions to take at every decision node you could ever find yourself at. For Player 2 there are just two pure strategies: Call or Fold. For Player 1 there are four: we denote them as AA, AK, KA, and KK, where the first component prescribes what to do when he has an ace, and the second component prescribes what to do when he has a king. For example, the strategy KA says "when you have an Ace, say you have a King (i.e., fold); and when you have a King, say you have an Ace (i.e., bet)."

Since Player 1 has four pure strategies and Player 2 has two, the game's strategic form is represented by a 4x2 bimatrix of strategy profiles and associated expected payoffs. The expected payoffs for Player 1 in each of the eight strategy profiles are easy to calculate:

<u>Player 1's Expected Payoff at each Strategy Profile</u>

	Holds an Ace		Holds a King		
(AA, Call):	(1/2) (2)	+	(1/2) (-2)	=	0
(AA, Fold):	(1/2) (1)	+	(1/2) (1)	=	1
(AK, Call):	(1/2) (2)	+	(1/2) (-1)	=	1/2
(AK, Fold):	(1/2) (1)	+	(1/2) (-1)	=	0
(KA, Call):	(1/2) (-1)	+	(1/2) (-2)	=	-3/2
(KA, Fold):	(1/2) (-1)	+	(1/2) (1)	=	0
(KK, Call):	(1/2) (-1)	+	(1/2) (-1)	=	-1
(KK, Fold):	(1/2) (-1)	+	(1/2) (-1)	=	-1

The Game's Strategic Form:

Player 2's Strategy

Player 1's Strategy		Call	Fold
	AA	**0 , 0**	**1 , -1**
	AK	**1/2 , -1/2**	**0 , 0**
	KA	**-3/2 , 3/2**	**0 , 0**
	KK	**-1 , 1**	**-1 , 1**

The first component of a strategy for Player 1 tells him what card to claim he has when he actually has an Ace.
The second component of a strategy for Player 1 tells him what card to claim he has when he actually has a King.

Dominated Strategies:

KK is strongly dominated by both AA and AK.

KA is strongly dominated by AA (and weakly dominated by AK).

In other words, when Player 1 holds an Ace, it is always better for him to say that he has an Ace than to fold.

The 2x2 Game:

After eliminating the dominated strategies KK and KA, we have a 2x2 strategic form game:

Player 2's Strategy

Player 1's Strategy		Call	Fold
	AA	**0 , 0**	**1 , -1**
	AK	**1/2 , -1/2**	**0 , 0**

Equilibrium:

Clearly there is no pure-strategy equilibrium. Since this is a zero-sum game, the equilibrium is for each player to play his minimax mixture (which is also his maximin mixture). Player 1's minimax mixture is (1/3, 2/3) on AA and AK, and Player 2's minimax mixture is (2/3, 1/3) on Call and Fold. The value of the game is 1/3 to Player 1 and –1/3 to Player 2. On average, then, we should expect Player 1 to win $1 for every three hands that are played.

A Description of Equilibrium Play

When the players are playing the equilibrium strategies Player 1 always bets (says "I have an Ace") when he has an Ace; when he has a King he "bluffs" one-third of the time, saying he has an Ace. Player 2 Calls two-thirds of the time when he gets to move – i.e., two-thirds of the time after Player 1 bets.

Comparison of Equilibrium with Player 1's "Honesty" Strategy

Recall that when Player 1 plays "honestly" he wins $1 (for sure) when he is dealt an Ace (which is half the time) and he loses $1 (for sure) when he is dealt a King (also half the time). So the "honesty" strategy yields him an expected value of zero.

If he instead uses his minimax mixture (and if Player 2 calls him two-thirds of the time), he still loses $1 when he is dealt a King (but now it's an *average* of $1, not a $1 loss every time). And he still always says he has an ace whenever he does indeed have one. But because he bluffs one-third of the time when he has a king (thereby inducing Player 2 to call him on two-thirds of his bets), *he now wins more on average when he has an Ace than he did before.* Moreover, *Player 2 can't do anything about this.* Indeed, because Player 1 is playing his minimax mixture, he'll win an average of $1/3 per play, no matter what Player 2 does! It's easy to check, for example, that this is what happens if Player 2 always Folds, and also if Player 2 always Calls.