Mixed Strategies: Some Examples, and Maximin Analysis

In the preceding lecture we analyzed the pursuit-and-evasion matches we had played in class. A critical strategic element of a pursuit-and-evasion situation is the need to be **unpredictable.**

Some further examples of strategic situations requiring unpredictability:

- Baseball: Pitcher vs. batter (any other baseball examples?)
- Basketball: Offensive player driving on a defender
- Soccer: Kicker vs. goalie
- Tennis: (a) Server vs. receiver
  (b) Down-the-line or crosscourt on a passing attempt
- Poker: Bluffing is necessary, and has to be unpredictable
- Placement of forces: Military battles and campaigns
- Wars of attrition: (strategies: when to give up and stop fighting)

  -- Animals competing for food, territory, mating opportunities
  -- Price wars
  -- Union vs. management in a strike
- Traffic control: authority vs. users
- Truth telling and fraud

Mixed Strategies: Maximin, the Best Worst Case

- In the preceding lecture we analyzed the winner-take-all match by analyzing the 2x2 point game.
- We verified that the Pursuer could ***assure*** himself of a 50% chance of winning the match.
- He can do that by using a mixed strategy that ***assures*** him a 2/9 chance of winning any point.
- The mixture that achieves this is **r = 2/3** on Left, and **1-r = 1/3** on Right. (**r** for *Row's mixture*)
- That mixture is the one that gives him the ***best*** **Worst Case**, a 2/9 expected payoff.
- This is also called his **maximin mixture:**

  it *maximizes* the *minimum* expected payoff the Evader can impose on him.

In our example, the Pursuer's expected payoff from choosing a mixture **r** was:

$\pi(r, L) = r(1/3) + (1-r)(0)$, Pursuer's Payoff(vs. Evader's Left choice)

$\pi(r, R) = r(0) + (1-r)(2/3)$, Pursuer's Payoff(vs. Evader's Right choice)

As we saw, the **r**-value that makes the minimum of these two payoffs the largest (i.e., the **r** that maximizes the minimum, the **maximin mixture**), is the mixture **r** that equates these two payoffs:

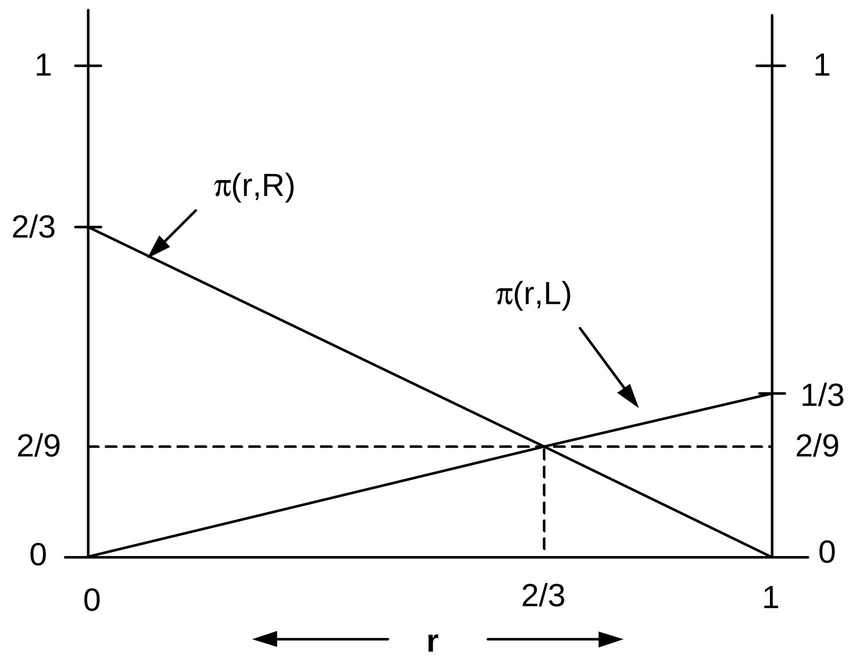
$$\pi(r, L) = \pi(r, R) .$$

i.e., $r(1/3) + (1-r)(0) = r(0) + (1-r)(2/3)$

i.e., $r = 2/3$.

Graph of the Row Player's Maximin Analysis

Graphs of π(r, L) and π(r, R):



(Vertical axis measures the Row player's expected payoff)

By choosing the mixture **r = 2/3** the Row player doesn't care what the Column player does: whether Column chooses Left or Right, the Row player's expected payoff (probability of winning the point, in our example) will be 2/9. And if the Column player chooses some *mixture* of Left and Right, the Row player's expected payoff will be a mixture of 2/9 and 2/9 -- still 2/9. (The expected payoff for Row when Column chooses a mixture c, i.e., π(r,c), would be a weighted average of the two depicted lines π(r,L) and π(r,R). But any such weighted-average line would of course pass through the intersection of the two depicted lines, so it would also yield expected payoff 2/9 at the Row mixture **r = 2/3**.)

But if the Row player chooses a mixture **r** different from 2/3, then the Column player could impose a Worst Case payoff of less than 2/9 on the Row player: for r < 2/3 this would happen if Column chooses Left; and for r > 2/3, this would happen if Column chooses Right.

Maximin Calculation in any 2x2 Game

Let's substitute arbitrary payoff numbers for the Row player into the 2x2 matrix:

		Evader L	Evader R
Pursuer	**L**	π_{LL}	π_{LR}
	R	π_{RL}	π_{RR}

The Row player's expected payoff from choosing a mixture **r** is:

$\pi(r, L) = r\,\pi_{LL} + (1-r)\,\pi_{RL}$, if the Column player were to choose Left

$\pi(r, R) = r\,\pi_{LR} + (1-r)\,\pi_{RR}$, if the Column player were to choose Right.

We know that the Row player's **maximin mixture** is the **r-**value that equates these two payoffs:

$$\pi(r, L) = \pi(r, R).$$

i.e., $$r\,\pi_{LL} + (1-r)\,\pi_{RL} = r\,\pi_{LR} + (1-r)\,\pi_{RR}$$

Solving this equation for **r** yields

$$\mathbf{r} = \frac{\pi_{RL} - \pi_{RR}}{(\pi_{RL} - \pi_{RR}) + (\pi_{LR} - \pi_{LL})} \quad \text{and} \quad \mathbf{1 - r} = \frac{\pi_{LR} - \pi_{LL}}{(\pi_{RL} - \pi_{RR}) + (\pi_{LR} - \pi_{LL})} .$$

Notice that in the expressions above for Row's mixture probabilities on Left and on Right:

The numerator for Left (i.e., for **r**) is the difference in the payoff entries if Row chooses Right

The numerator for Right (i.e., for **1-r**) is the difference in the payoff entries if Row chooses Left

The denominator is the sum of the two payoff differences.

Of course, if the payoff numbers in the matrix are the **Column** player's payoffs, then we can do the same analysis for the **Column player's maximin mixture**:

The Column player's expected payoff from choosing a mixture **c** is:

$\pi(L, c) = c\,\pi_{LL} + (1-c)\,\pi_{LR}$, if the Row player were to choose Left

$\pi(R, c) = c\,\pi_{RL} + (1-c)\,\pi_{RR}$, if the Row player were to choose Right.

Setting the two payoffs to be equal and solving for **c** yields

$$\mathbf{c} = \frac{\pi_{LR} - \pi_{RR}}{(\pi_{LR} - \pi_{RR}) + (\pi_{RL} - \pi_{LL})} \quad \text{and} \quad \mathbf{1 - c} = \frac{\pi_{RL} - \pi_{LL}}{(\pi_{LR} - \pi_{RR}) + (\pi_{RL} - \pi_{LL})} .$$

For the Column player:

The numerator for Left (i.e., for **c**) is the difference in the payoff entries if he chooses Right

The numerator for Right (i.e., for **1-c**) is the difference in the payoff entries if he chooses Left

The denominator is the sum of the two payoff differences.

The Column Player's Maximin Mixture in the Pursuit-Evasion Game

Evader

Pursuer		**L**	**R**	**c=1/2**	**c=3/4**	**Any c**	**c=2/3**
	L	**2/3**	**1**	**5/6**	**3/4**	**1 - 1/3 c**	**7/9**
	R	**1**	**1/3**	**2/3**	**5/6**	**1/3 + 2/3 c**	**7/9**

$$\pi\,(\,L, c\,) = c\,\pi_{LL} + (1-c)\,\pi_{LR} = (2/3)\,c + 1\,(1-c) = 1 - (1/3)\,c$$

$$\pi\,(\,R, c\,) = c\,\pi_{RL} + (1-c)\,\pi_{RR} = 1\,c + (1/3)\,(1-c) = 1/3 + (2/3)\,c$$

The **c-**value that equates the two payoffs:

$$\pi\,(\,L, c\,) = \pi\,(\,R, c\,)$$

i.e., $1 - (1/3)\,c = 1/3 + (2/3)\,c$

i.e., $\mathbf{c = 2/3}$ **.**

The Column player's **maximin value**: $1 - (1/3)\,(2/3) = 1/3 + (2/3)\,(2/3) = 7/9$.