Mixed Strategies: Outline of the Lecture

1. Some questions about the winner-take-all match/game we played.
2. Predictions about play in the game (with accumulated winnings, not pay-per-play):

   http://www.u.arizona.edu/~mwalker/RSWPredictions.pdf
3. How did all of you play? How did people play in past matches? (Look at spreadsheets)

   http://www.u.arizona.edu/~mwalker/MatchPlaySummary.pdf
4. Was the game fair? How can we determine the answer to this question?
5. Probabilities of winning the match: binomial model (points are independent)
6. Is there any way to assure yourself of a 50%, or better, chance of winning the match?
7. Is there any way to assure yourself of a 2/9, or better, chance of winning each point?
8. The 2x2 game played for each point.
9. The game has no Nash equilibrium (in *pure strategies*), and no strategies are dominated.
10. **You have to be unpredictable!**   But what does that mean, in practice?
11. Using a *mixed strategy*.   Adding mixed strategies to the 2x2 game.
12. The Worst Case: your lowest possible probability of winning the current point
13. Maximin (the *best* Worst Case): ***Assurance*** of a minimum probability of winning.

Mixed Strategies:
Analyzing the Winner-Take-All Matches

Some questions about the match/game:

- Is it fair? Does each player have the same chance of winning?
- What's a good way to play the game?
- Is there a *best* way to play the game?
- How do people *actually* play in this game?

Is the game fair?

- Can we answer this by looking at how many matches were won by, say, the Pursuers?
  - -- No: small sample. And even with a large sample, players may have played badly
  - -- We need to *figure out* the answer directly, analytically
- What is the probability the Pursuer will win the match – i.e., that he'll win 22 or more points?
- What does that probability depend on?
- What is the probability the Evader will win 76 or more points?
- What is the probability the Pursuer (or Evader) will win a *given* point?
- Can the Pursuer assure himself a 2/9 probability of winning each point? A larger probability?
- Can the Evader assure himself a 7/9 probability of winning each point? A larger probability?

Your Probability of Winning the Match

Suppose p is the probability the Pursuer will win *any given point*. What is the probability he'll win at least 22 points? We are modeling the winning of points as a *binomial process*:

- Each point is independent of the other points
- Probability of a "success:" p
- Probability of s successes in n trials: Pr($s, n ; p$) , the binomial probability distribution
- It's easy to evaluate binomial probabilities in Excel:

Syntax:

BINOMDIST(number_s, trials, probability_s, cumulative)

Number_s is the number of successes in trials.

Trials is the number of independent trials.

Probability_s is the probability of success on each trial.

Cumulative is a logical value that determines the form of the function. If cumulative is TRUE, then BINOMDIST returns the cumulative distribution function, which is the probability that there are at most *number_s* successes; if FALSE, it returns the probability mass function, which is the probability that there are *number_s* successes.

Is the Game Fair?

We look at the Excel spreadsheet containing the cumulative binomial probabilities (following page).

We see that if the outcomes of the points are a binomial process, then the Pursuer will have a 50% chance to win 22 or more points *if his probability of winning a given point is .22222* -- about 2/9.

If $p > 2/9$, then the Pursuer has a greater than 50% chance of winning the match – the match is biased in favor of the Pursuer. And small variations in p have a very large effect.

If $p < 2/9$, then the Pursuer has a less than 50% chance of winning the match – the match is biased in favor of the Evader. And again, small variations in p have a very large effect.

So now we need to answer these questions:

- Can the Pursuer assure himself a 2/9 chance of winning on each point?
- Can the Pursuer assure himself a *greater* than 2/9 chance of winning on each point?
- Can the Evader assure himself a 7/9 chance (or greater) of winning on each point?

The Probability that the Row Player Wins At Least s Times in 97 Plays

(p is the probability that Row wins any given point)

p	s = 15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0.15	.49	.38	.28	.20	.13	.08	.05	.03	.02	.01	.00	.00	.00	.00	.00	.00
0.16	.60	.49	.38	.28	.20	.14	.09	.05	.03	.02	.01	.00	.00	.00	.00	.00
0.17	.70	.59	.49	.38	.29	.20	.14	.09	.06	.03	.02	.01	.01	.00	.00	.00
0.18	.78	.69	.59	.48	.38	.29	.21	.14	.09	.06	.04	.02	.01	.01	.00	.00
0.19	.85	.77	.68	.59	.48	.38	.29	.21	.15	.10	.06	.04	.02	.01	.01	.00
0.20	.90	.84	.77	.68	.58	.48	.38	.29	.21	.15	.10	.06	.04	.02	.01	.01
0.21	.93	.89	.83	.76	.67	.58	.48	.38	.29	.21	.15	.10	.07	.04	.02	.01
0.22	.96	.93	.88	.83	.75	.67	.57	.48	.38	.29	.22	.15	.11	.07	.04	.03
0.23	.97	.95	.92	.88	.82	.75	.66	.57	.47	.38	.29	.22	.16	.11	.07	.04
0.24	.99	.97	.95	.92	.87	.81	.74	.66	.57	.47	.38	.29	.22	.16	.11	.07
0.25	.99	.98	.97	.95	.91	.87	.81	.74	.65	.56	.47	.38	.29	.22	.16	.11
0.26	1.00	.99	.98	.97	.94	.91	.86	.80	.73	.65	.56	.47	.38	.29	.22	.16
0.27	1.00	.99	.99	.98	.96	.94	.91	.86	.80	.73	.64	.56	.46	.38	.29	.22
0.28	1.00	1.00	.99	.99	.98	.96	.94	.90	.85	.79	.72	.64	.55	.46	.38	.29
0.29	1.00	1.00	1.00	.99	.99	.98	.96	.93	.90	.85	.79	.72	.64	.55	.46	.37
0.30	1.00	1.00	1.00	1.00	.99	.99	.97	.96	.93	.89	.85	.79	.71	.63	.55	.46
0.31	1.00	1.00	1.00	1.00	1.00	.99	.98	.97	.95	.93	.89	.84	.78	.71	.63	.54
0.22222	.96	.93	.89	.84	.77	.69	.59	.50	.40	.31	.23	.17	.12	.08	.05	.03

	If p is .22222:	
Total # of Plays	Pr of 22+	Pr of 23+
95	.453	.358
96	.475	.379
97	.496	.401
98	.518	.422
99	.539	.443
100	.560	.465
101	.581	.486
102	.601	.507

The 2x2 "Point Game"

We are assuming that each point is played independently – i.e., at each point, a player's sole objective is to win that point.

The game played for each point is the following 2x2 game, in which a player's payoff in any cell (i.e., for any profile of choices) is the probability he will win the point:

Evader

		L	**R**
Pursuer	**L**	**1/3, 2/3**	**0, 1**
	R	**0, 1**	**2/3, 1/3**

The game has no dominated strategies, and it has no Nash equilibrium in *pure strategies* – i.e., none of the four strategy profiles LL, LR, RL, or RR is a Nash equilibrium.

A player has to be ***unpredictable*** in his play. A foolproof way to be unpredictable is to ***randomize*** among your available strategies. For example, you could flip a coin, and then play Left if it comes up Heads, and play Right if it comes up Tails. If a player considers such ***mixtures*** among his strategies – i.e., ***mixed strategies*** – it expands the set of strategies available to him.

Mixed Strategies and the "Worst Case"

Let's add the mixed strategy "Play Left with 50% probability, play Right with 50% probability" to the Pursuer's arsenal of strategies, and for each of his (now) three strategies, let's determine the Worst Case that the Evader can impose by choosing either Left or Right:

		Evader L	Evader R	Worst Case	Evader Choice
Pursuer	L	1/3	0	0	R
	R	0	2/3	0	L
	50-50	1/6	1/3	1/6	L

The Worst Case under the 50-50 mixed strategy is better than under either of the Pursuer's pure strategies. But it looks as if the Pursuer could make his Worst Case even better by using a mixture that increases the "1/6" entry, by placing more probability on playing Left. This is done on the following page.

Getting a Better Worst Case with Other Mixtures

From now on, let's keep track of the various possible mixed strategies for the Pursuer (the Row player) by using **r** to denote a mixture that plays Left with probability **r** and plays Right with probability **1-r**. And let's see what the Worst Case would be if the Pursuer uses the mixture **r = 3/4** – i.e., Left with probability 3/4 and Right with probability 1/4.

		Evader L	Evader R	Worst Case	Evader Choice
Pursuer	L	1/3	0	0	R
	R	0	2/3	0	L
	r=1/2	1/6	1/3	1/6	L
	r=3/4	1/4	1/6	1/6	R

How did we get those two entries, 1/4 and 1/6 ? Remember, each cell entry is the probability the Pursuer will win the point if that cell's strategies are chosen. So we have, for the mixture **r = 3/4:**

$$\text{Payoff (vs. L)} = r\,(1/3) + (1-r)\,(0) = (3/4)(1/3) + (1/4)(0) = 1/4 + 0 = 1/4 \quad (1)$$

$$\text{Payoff (vs. R)} = r\,(0) + (1-r)\,(2/3) = (3/4)(0) + (1/4)(2/3) = 0 + 2/12 = 1/6 \quad (2)$$

Finding the Best Worst Case

The mixture **r = 3/4** clearly placed *too much* weight on playing Left: it pushed the Pursuer's payoff when the Evader plays Right *below* his payoff against Left:

$$\text{Payoff (vs. R)} < \text{Payoff (vs. L)} .$$

It's clear that as we increase **r** (the mixture weight on Left), we *increase* Payoff (vs. L) and we *decrease* Payoff (vs. R). (This is quite intuitive: the Pursuer wants to choose the same direction as the Evader.) So we'll find the *best* Worst Case by choosing an **r** that *equates* these two payoffs:

$$\text{Payoff (vs. L)} = \text{Payoff (vs. R)}$$

i.e., $r\,(1/3) + (1-r)\,(0) = r\,(0) + (1-r)\,(2/3)$, from equations (1) and (2)

i.e., $(1/3)\,r = (2/3) - (2/3)\,r$

i.e., $r = 2/3$.

Notice that with the mixture **r = 2/3** , each of the two payoffs is 2/9:

$$\text{Payoff (vs. L)} = \text{Payoff (vs. R)} = 2/9.$$

Let's see how this looks when we add the mixed strategy **r = 2/3** to the Pursuer's payoff table, on the following page.

The <u>Best</u> Worst Case

Adding the mixed strategy **r = 2/3** to the Pursuer's payoff table:

		Evader L	Evader R	Worst Case	Evader Choice
	L	1/3	0	0	R
	R	0	2/3	0	L
Pursuer	r=1/2	1/6	1/3	1/6	L
	r=3/4	1/4	1/6	1/6	R
	r=2/3	2/9	2/9	2/9	L or R

It's clear that this mixed strategy is the one that gives the Pursuer his *best* Worst Case: we know that if we *increase* **r** that will decrease the Payoff (vs. R), making the Worst Case winning probability less than 2/9; and if we *decrease* **r** that will decrease the Payoff (vs. L), also making the Worst Case winning probability less than 2/9.

In other words, by using the mixed strategy **r = 2/3** the Pursuer can ***assure*** himself that he will have a 2/9 chance of winning the current point. And we've already seen that if he can do that, then he will assure himself a 50% chance of winning the match.