Norms and Metrics,
Normed Vector Spaces and Metric Spaces

We're going to develop generalizations of the ideas of length (or magnitude) and distance. We'll generalize from Euclidean spaces to more general spaces, such as spaces of functions. We begin with the familiar notions of magnitude and distance on the real line.

Definition: The **absolute value** of a number $x \in \mathbb{R}$ is $|x| := \max\{x, -x\}$, which we also call the size, magnitude, length, or norm of x.

Remark: The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}_+$ has the following properties:

(N1) $\forall x \in \mathbb{R} : |x| \geqq 0$;

(N2) $\forall x \in \mathbb{R} : |x| = 0 \;\Leftrightarrow\; x = 0$;

(N3) $\forall x, y \in \mathbb{R} : |x + y| \leqq |x| + |y|$;

(N4) $\forall \alpha \in \mathbb{R}, x \in \mathbb{R} : |\alpha x| = |\alpha||x|$.

Our second familiar idea is the distance between two points on the line, *i.e.*, between two real numbers:

Definition: The **distance** between two numbers x and x' in $\mathbb{R}$ is the absolute value of their difference: $d(x, x') := |x - x'|$.

Remark: The distance function $d : \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}_+$ has the following properties:

(D1) $d(x, x') \geqq 0$;

(D2) $d(x, x') = 0 \;\Leftrightarrow\; x = x'$;

(D3) $d(x, x') = d(x', x)$;

(D4) $d(x, x'') \leqq d(x, x') + d(x', x'')$, the **triangle inequality**.

In Euclidean space the length of a vector, or equivalently the distance between a point and the origin, is its *norm*, and just as in $\mathbb{R}$, the distance between two points is the norm of their difference:

Definitions: The **Euclidean norm** of an element $x \in \mathbb{R}^n$ is the number

$$\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The **Euclidean distance** between two points $x, x' \in \mathbb{R}^n$ is

$$d(x, x') := ||x - x'||.$$

Remark: The Euclidean norm function $||\cdot||:\mathbb{R}^n \to \mathbb{R}_+$ has the properties (N1) - (N4); the Euclidean distance function $d:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}_+$ has the properties (D1) - (D4).

Definition: Let V be a vector space. A function $||\cdot||:V\to\mathbb{R}_+$ is a **norm** on V if it satisfies (N1) - (N4). A vector space together with a norm is called a **normed vector space**.

Definition: Let X be a set. A **metric** on X is a function $d:X\times X\to\mathbb{R}_+$ that satisfies (D1) - (D4). The pair (X,d) is called a **metric space**.

Remark: If $||\cdot||$ is a norm on a vector space V, then the function $d:V\times V\to\mathbb{R}_+$ defined by $d(x,x'):=||x-x'||$ is a metric on V

In other words, a normed vector space is automatically a metric space, by defining the metric in terms of the norm in the natural way. But a metric space may have no algebraic (vector) structure — *i.e.*, it may not be a vector space — so the concept of a metric space is a generalization of the concept of a normed vector space.

In each of the following examples you should verify that d is a metric by verifying that it satisfies each of the four conditions (D1) to (D4). For the norms on $\mathbb{R}^n$, you should draw the set of all points in $\mathbb{R}^2$ that satisfy $||x||=1$.

Example 1: In $\mathbb{R}^n$ define $||x||_\infty := \max\{|x_1|,\ldots,|x_n|\}$ and define $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ as $d(x,x')=||x-x'||_\infty=\max\{|x_1-x_1'|,\ldots,|x_n-x_n'|\}$. This is called the *max*-norm or *sup*-norm.

Example 2: In $\mathbb{R}^n$ define $||x||_1 := \sum_{i=1}^n |x_i|$ and $d(x,x')=||x-x'||_1=\sum_{i=1}^n |x_i-x_i'|$. This is called the city block, Manhattan, or taxicab norm.

Example 3: Let $X=C[0,1]=\{f:[0,1]\to\mathbb{R} \mid f \text{ is continuous }\}$. Define $||\cdot||:C[0,1]\to\mathbb{R}$ by $||f||:=\max\{|f(x)| \mid x\in[0,1]\}$. Note that this is well-defined as a consequence of the Weierstrass Theorem, which says that a continuous real function on a closed interval attains a maximum. We will introduce the Weierstrass Theorem formally a little later in the course, and generalize it to compact sets in metric spaces. Also note that $C[0,1]$ is a vector space under natural definitions of vector addition and scalar multiplication:

$$(f+g)(x):=f(x)+g(x) \quad\text{and}\quad (\alpha f)(x):=\alpha f(x).$$

Exercise: Show that the set $C[0,1]$ with the operations of vector addition and scalar multiplication defined in Example 3 is a vector space.

Example 4: Let $X = \{a, b\}$ or any other finite set. Define $d : X \times X \to \mathbb{R}_+$ as follows:

$$d(x, x') := \begin{cases} 0 & \text{if } x = x' \\ 1 & \text{otherwise .} \end{cases}$$

Then d is a metric on X. This is an example of a metric space that is not a normed vector space: there is no way to define vector addition or scalar multiplication for a finite set.

Example 5: Let p be a real number satisfying $p \geqq 1$, and define $||x||_p$ on $\mathbb{R}^n_+$ by

$$\|x\|_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}.$$

This is called the ℓ_p norm. Note that the Euclidean norm is the ℓ_2-norm, the city block norm is the ℓ_1-norm, and the *sup*-norm is the ℓ_∞-norm.

Remark: If (X, d) is a metric space and S is a subset of X, then (S, d) is a metric space.

Example 6: Let V be a normed vector space — for example, $\mathbb{R}^2$ with the Euclidean norm. Let C be the unit circle $\{x \in V \mid ||x|| = 1\}$. This is another example of a metric space that is not a normed vector space: V is a metric space, using the metric defined from $|| \cdot ||$, and therefore, according to the above remark, so is C; but C is not a vector space, so it is not a normed vector space. C could be replaced here by any subset of V that is not a vector subspace of V — *i.e.*, any subset that's not closed under vector addition and scalar multiplication.