The Bolzano-Weierstrass Property and Compactness

We know that not all sequences converge. In fact, the ones that do converge are just the "very good" ones. But even "very good" sequences may not converge. Cauchy sequences, for example, are very good; they're not really any different than convergent sequences: a Cauchy sequence actually does converge in "good" spaces (*i.e.*, complete spaces), and fails to converge only if the point that "should be" its limit is not in the space — *i.e.*, it fails to converge because the *space* is not good (is incomplete), not because the sequence is bad.

Now let's ask about a weaker property of sequences: which sequences have cluster points? Equivalently, when does a sequence have a subsequence that converges? Consider the alternating real sequence $\{1, -1, 1, -1, 1, \ldots\}$. The sequence certainly doesn't converge, but it has subsequences that do, such as $\{1, 1, 1, \ldots\}$. We now study an easy-to-check condition that guarantees that a sequence in $\mathbb{R}$ or $\mathbb{R}^n$ has a convergent subsequence: every *bounded* sequence in $\mathbb{R}^n$ has a Cauchy subsequence, a subsequence that therefore converges in $\mathbb{R}^n$. That's the content of the Bolzano-Weierstrass Theorem. (We're assuming throughout this section that $\mathbb{R}^n$ is endowed with a norm; we've already seen that *which* norm we use in $\mathbb{R}^n$ has no effect on convergence — *i.e.*, on which sequences converge.)

The Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence and without loss of generality assume that every term of the sequence lies in the interval $[0,1]$. Divide $[0,1]$ into two intervals, $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. (Note: this is not a partition of $[0,1]$.) At least one of the halves contains infinitely many terms of $\{x_n\}$; denote that interval by I_1, which has length $\frac{1}{2}$, and let $\{x_{n'}\}$ be the subsequence of $\{x_n\}$ consisting of every term that lies in I_1.

Now divide I_1 into two halves, each of length $\frac{1}{4} = (\frac{1}{2})^2$, at least one of which contains infinitely many terms of the (sub)sequence $\{x_{n'}\}$, and denote that half by I_2. Let $\{x_{n''}\}$ be the subsequence of $\{x_{n'}\}$ consisting of all of the terms that lie in I_2. Continuing in this way, we construct a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \ldots$, where the length of I_n is $(\frac{1}{2})^n$, and each interval contains an infinite number of terms of the original sequence $\{x_n\}$. Finally, we construct a subsequence $\{z_n\}$ of $\{x_n\}$ made up of one term from each interval I_n. This subsequence is clearly Cauchy: $\forall$N: $m, n > N \Rightarrow |z_m - z_n| < (\frac{1}{2})^N$. Therefore the subsequence $\{z_n\}$ converges, according to the Cauchy-sequence version of the Completeness Axiom. $\|$

The Bolzano-Weierstrass Theorem is true in $\mathbb{R}^n$ as well:

The Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof: Let $\{x^m\}$ be a bounded sequence in $\mathbb{R}^n$. (We use superscripts to denote the terms of the sequence, because we're going to use subscripts to denote the components of points in $\mathbb{R}^n$.) The sequence $\{x_1^m\}$ of first components of the terms of $\{x^m\}$ is a bounded real sequence, which has a convergent subsequence $\{x_1^{m_k}\}$, according to the B-W Theorem in $\mathbb{R}$. Let $\{x^{m_k}\}$ be the corresponding subsequence of $\{x^m\}$. Then the sequence $\{x_2^{m_k}\}$ of second components of $\{x^{m_k}\}$ is a bounded sequence of real numbers, so it too has a convergent subsequence, and we again have a corresponding subsequence of $\{x^{m_k}\}$ (and therefore of $\{x^m\}$), in which the sequences of first and second components both converge. Continuing for n iterations, we end up with a subsequence $\{z^m\}$ of $\{x^m\}$ in which the sequences of first, second, ..., nth components all converge, and therefore the subsequence $\{z^m\}$ itself converges in $\mathbb{R}^n$. $\|$

It's elementary to show that the following form of the B-W Theorem is equivalent to the one we've just proved:

The Bolzano-Weierstrass Theorem: Every sequence in a closed and bounded set S in $\mathbb{R}^n$ has a convergent subsequence (which converges to a point in S).

Proof: Every sequence in a closed and bounded subset is bounded, so it has a convergent subsequence, which converges to a point in the set, because the set is closed. $\|$

Conversely, every bounded sequence is in a closed and bounded set, so it has a convergent subsequence.

Subsets of $\mathbb{R}^n$ that are both closed and bounded are so important that we give them their own name: a closed and bounded subset of $\mathbb{R}^n$ is said to be **compact**. And in any metric space, the sets in which all bounded sequences have convergent subsequences are so important that we give that property of sets its own name as well:

Definition: A set S in a metric space has the **Bolzano-Weierstrass Property** if every sequence in S has a convergent subsequence — *i.e.*, has a subsequence that converges to a point in S.

The B-W Theorem states that closed and bounded (*i.e.*, compact) sets in $\mathbb{R}^n$ have the B-W Property. We can also prove the converse of the B-W Theorem, that any set in $\mathbb{R}^n$ with the B-W Property is closed and bounded (compact). See Theorem 29.6 of Simon & Blume.

In other words, the compact sets in $\mathbb{R}^n$ are *characterized* by the Bolzano-Weierstrass Property.

So how do things work in general metric spaces? Are compact sets characterized by B-W?

What is the definition of a compact set in a metric space?

Let's say, tentatively, that it's still defined as a closed and bounded set.

Are closed and bounded sets characterized by the B-W Property in metric spaces?

Example: Let (X, d) be an infinite set with the discrete metric: $x \neq x' \Rightarrow d(x, x') = 1$. (For example, let $X = \mathbb{N} = \{1, 2, \ldots\}$.) Of course the space is closed. And the space is bounded: let x^* be any element of X; every element $x \in X$ except x^* is at distance $d(x, x^*) = 1$ from x^*. Let $\{x_n\}$ be any sequence of distinct points in X; the sequence is bounded. Moreover, it does not have a convergent subsequence (it consists of distinct points, all of them equidistant from one another). So closed and bounded sets in a metric space don't necessarily have the B-W Property — in this respect, closed and bounded sets in some metric spaces will behave very differently than compact sets in $\mathbb{R}^n$. **Exercise:** Is the sequence described in the example Cauchy? Describe all Cauchy sequences and all convergent sequences in this metric space.

The essential feature of compact sets in $\mathbb{R}^n$ is that they have the B-W Property. Lots of other properties of compact sets follow from that — for example, the Weierstrass Theorem, that a continuous real-valued function on a compact set attains a maximum and a minimum. But here we've seen that closed and bounded sets in an arbitrary metric space may not have the B-W property; therefore we don't want to call them compact. Instead, we simply *define* compact sets to be the ones that have the B-W Property.

Definition: A metric space is **compact** if it has the B-W Property.

Let's review: In $\mathbb{R}^n$ we called the closed and bounded sets compact, and they were characterized by the B-W Property. In metric spaces we have definitions of closed sets and bounded sets, but closed and bounded sets don't necessarily have the B-W Property. So we *defined* compact sets to be the ones that have the B-W Property — so in $\mathbb{R}^n$ the compact sets are still the closed and bounded ones, and now in *all* metric spaces the compact sets (as in $\mathbb{R}^n$) are precisely the ones with the B-W Property.

The following two theorems are easy to prove:

Theorem: Let S be a compact set in a metric space. Then
(a) S is closed; (b) S is bounded; (c) S is complete.

Theorem: A closed subset of a compact metric space is compact.

Now we can easily prove the Weierstrass Theorem; in fact, we can prove the following generalized form of the Weierstrass Theorem, which says that continuous functions preserve compactness.

Theorem: Let $f : X \to Y$. If X is compact and f is continuous, then $f(X)$ is compact.

Proof: We must show that $f(X)$ has the B-W Property. Let $\{y_n\}$ be a sequence in $f(X)$; we must show that $\{y_n\}$ has a convergent subsequence. For each $n \in \mathbb{N}$, let $x_n \in X$ be such that $f(x_n) = y_n$ (which we can do because $y_n \in f(X)$). Since X is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $\overline{x} \in X$. Since f is continuous, $\{f(x_{n_k})\}$ converges to $f(\overline{x}) \in Y$. Since $\overline{x} \in X$, we have $f(\overline{x}) \in f(X)$. $\|$

Corollary (The Weierstrass Theorem): A continuous real-valued function on a compact subset S of a metric space attains a maximum and a minimum on S.

Proof: $f(S)$ is a compact subset of $\mathbb{R}$, *i.e.*, a closed and bounded subset of $\mathbb{R}$. Since $f(S)$ is a bounded subset of $\mathbb{R}$, it has both a least upper bound M and a greatest lower bound m; and since $f(S)$ is closed, it contains m and M. Therefore $m = \min f(S)$ and $M = \max f(S)$. $\|$

Exercise: In the example on the preceding page, (X, d) is an infinite discrete metric space. Which subsets of X are compact? Which subsets are closed and bounded? Which subsets are open? Let $X = \mathbb{Z}$, the set of all integers, and let $f : \mathbb{Z} \to \mathbb{R}$ be the real-valued function $f(x) = x$. Does f attain a maximum or a minimum on $\mathbb{Z}$? Is f continuous? On which subsets of $\mathbb{Z}$ does f attain a maximum or a minimum?

Note: You may also see a definition that says a compact set is one that has the Heine-Borel Property — every open cover has a finite subcover. Just as "closed and bounded" didn't get us what we wanted when we went from $\mathbb{R}^n$ to a metric space, the B-W Property doesn't get us what we want when we go to a topological space (a space where open and closed sets are the defining concepts but the space may not have a metric structure).