Open and Closed Sets

Definition: A subset S of a metric space (X, d) is **open** if it contains an open ball about each of its points — *i.e.*, if

$$\forall x \in S : \exists \epsilon > 0 : B(x, \epsilon) \subseteq S. \tag{1}$$

Theorem:

($\mathcal{O}1$) $\emptyset$ and X are open sets.

($\mathcal{O}2$) If $S_1, S_2, \ldots, S_n$ are open sets, then $\cap_{i=1}^n S_i$ is an open set.

($\mathcal{O}3$) Let A be an arbitrary set. If S_α is an open set for each $\alpha \in A$, then $\cup_{\alpha \in A} S_\alpha$ is an open set. In other words, the union of any collection of open sets is open. [Note that A can be any set, not necessarily, or even typically, a subset of X.]

Proof:

($\mathcal{O}1$) $\emptyset$ is open because the condition (1) is vacuously satisfied: there is no $x \in \emptyset$. X is open because any ball is by definition a subset of X.

($\mathcal{O}2$) Let S_i be an open set for $i = 1, \ldots, n$, and let $x \in \cap_{i=1}^n S_i$. We must find an $\epsilon > 0$ for which $B(x, \epsilon) \subseteq \cap_{i=1}^n S_i$. For each i there is an ϵ_i such that $B(x, \epsilon_i) \subseteq S_i$, because each S_i is open. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $0 < \epsilon \leqq \epsilon_i$ for each i; therefore $B(x, \epsilon) \subseteq B(x, \epsilon_i) \subseteq S_i$ for each i — *i.e.*, $B(x, \epsilon) \subseteq \cap_{i=1}^n S_i$.

($\mathcal{O}3$) Let $x \in \cup_{\alpha \in A} S_\alpha$; we must find an $\epsilon > 0$ for which $B(x, \epsilon) \subseteq \cup_{\alpha \in A} S_\alpha$. Since $x \in \cup_{\alpha \in A} S_\alpha$, we have $x \in S_{\bar{\alpha}}$ for some $\bar{\alpha}$. Since $S_{\bar{\alpha}}$ is open, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S_{\bar{\alpha}} \subseteq \cup_{\alpha \in A} S_\alpha$. ■

Definition: A subset S of a metric space (X, d) is **closed** if it is the complement of an open set.

Theorem:

($\mathcal{C}1$) $\emptyset$ and X are closed sets.

($\mathcal{C}2$) If $S_1, S_2, \ldots, S_n$ are closed sets, then $\cup_{i=1}^n S_i$ is a closed set.

($\mathcal{C}3$) Let A be an arbitrary set. If S_α is a closed set for each $\alpha \in A$, then $\cap_{\alpha \in A} S_\alpha$ is a closed set. In other words, the intersection of any collection of closed sets is closed.

Proof:

($\mathcal{C}1$) follows directly from ($\mathcal{O}1$).

($\mathcal{C}2$) and ($\mathcal{C}3$) follow from ($\mathcal{O}2$) and ($\mathcal{O}3$) by De Morgan's Laws. ■

Exercise: Use De Morgan's Laws to establish ($\mathcal{C}2$) and ($\mathcal{C}3$).

Definition: A **limit point** of a set S in a metric space (X, d) is an element $\bar{x} \in X$ for which there is a sequence in $S \setminus \{\overline{x}\}$ that converges to $\bar{x}$ — *i.e.*, a sequence in S, none of whose terms is $\overline{x}$, that converges to $\overline{x}$. Limit points are also called **accumulation points** of S or **cluster points** of S.

Remark: $\overline{x}$ is a limit point of S if and only if every neighborhood of $\overline{x}$ contains a point in $S \setminus \{\overline{x}\}$; equivalently, if and only if every neighborhood of $\overline{x}$ contains an infinite number of points in S.

Proof of the above remark is an exercise. To prove that every neighborhood of a limit point $\overline{x}$ contains an infinite number of points, you may find it useful to invoke the Well-Ordering Property of the set $\mathbb{N}$ of natural numbers:

Definition: A totally ordered set $(X, \leqq)$ has the Well-Ordering Property (or is a well-ordered set) if every nonempty subset of X has a first (*i.e.*, smallest) element.

The Well-Ordering Principle: The set $\mathbb{N}$ of natural numbers with its usual order is well-ordered.

In the following theorem, limit points provide an important characterization of closed sets. As always with characterizations, this characterization is an alternative definition of a closed set. In fact, many people actually use this as the definition of a closed set, and then the definition we're using, given above, becomes a theorem that provides a characterization of closed sets as complements of open sets.

Theorem: A set is closed if and only if it contains all its limit points.

Proof:

($\Rightarrow$): Let S be a closed set, and let $\{x_n\}$ be a sequence in S (*i.e.*, $\forall n \in \mathbb{N} : x_n \in S$) that converges to $\bar{x} \in X$. We must show that $\bar{x} \in S$. Suppose not — *i.e.*, $\bar{x} \in S^c$. Since S^c is open, there is an $\epsilon > 0$ for which $B(\bar{x}, \epsilon) \subseteq S^c$. Since $\{x_n\} \to \bar{x}$, let $\bar{n}$ be such that $n > \bar{n} \Rightarrow d(x_n, \bar{x}) < \epsilon$. Then for $n > \bar{n}$ we have both $x_n \in S$ and $x_n \in B(\bar{x}, \epsilon) \subseteq S^c$, a contradiction.

($\Leftarrow$): Suppose S is not closed. We must show that S does not contain all its limit points. Since S is not closed, S^c is not open. Therefore there is at least one element $\bar{x}$ of S^c such that every ball $B(\bar{x}, \epsilon)$ contains at least one element of $(S^c)^c = S$. For every $n \in \mathbb{N}$, let $x_n \in B(\bar{x}, \frac{1}{n}) \cap S$. Then we have a sequence $\{x_n\}$ in S which converges to $\bar{x} \notin S$ — *i.e.*, $\bar{x}$ is a limit point of S but is not in S, so S does not contain all its limit points. ■

Example 1: For each $n \in \mathbb{N}$, let S_n be the open set $(-\frac{1}{n}, \frac{1}{n}) \subseteq \mathbb{R}$. Then $\cap_{n=1}^{\infty} S_n = \{0\}$, which is not open. This is a counterexample which shows that $(\mathcal{O}2)$ would not necessarily hold if the collection weren't finite.

Example 2: For each $n \in \mathbb{N}$, let S_n be the closed set $[\frac{1}{n}, \frac{n-1}{n}] \subseteq \mathbb{R}$. Then $\cup_{n=1}^{\infty} S_n = (0,1)$, which is not closed. This is a counterexample which shows that $(\mathcal{C}2)$ would not necessarily hold if the collection weren't finite.

Example 3: $\mathbb{R}^n_{++}$ is open. This can be shown directly, by finding an appropriate $\epsilon > 0$ for each $x \in \mathbb{R}$. Alternatively, one could show that for each $i = 1, \ldots, n$ the set $S_i = \{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0\}$ is open, and then invoke $(\mathcal{O}2)$ for the set $\mathbb{R}^n_{++} = \cap_{i=1}^n S_i$.

Example 4: The union of all open subsets of $\mathbb{R}^n_+$ is an open set, according to $(\mathcal{O}3)$. Note that this set is $\mathbb{R}^n_{++}$. This is therefore a third way to show that $\mathbb{R}^n_{++}$ is an open set.

Exercise: Is ℓ^{∞}_{++} an open subset of ℓ^{∞}? Prove that your answer is correct.

Example 5: Generalizing Example 4, let G be any subset of (X, d) and let $\mathring{G}$ be the union of all open subsets of G. According to $(\mathcal{O}3)$, $\mathring{G}$ is an open set. $\mathring{G}$ is clearly the "largest" open subset of G, in the sense that (i) $\mathring{G}$ is itself an open subset of G, and (ii) every open subset of G is a subset of $\mathring{G}$ — *i.e.*. $\mathring{G}$ contains every open subset of G, which we could state informally as $\mathring{G}$ is "at least as large" as every other open subset of G. We call the set $\mathring{G}$ the **interior** of G, also denoted *int* G.

Example 6: Doing the same thing for closed sets, let G be any subset of (X, d) and let $\overline{G}$ be the intersection of all closed sets that contain G. According to $(\mathcal{C}3)$, $\overline{G}$ is a closed set. It is the "smallest" closed set containing G as a subset, in the sense that (i) $\overline{G}$ is itself a closed set containing G, and (ii) every closed set containing G as a subset also contains $\overline{G}$ as a subset — every other closed set containing G is "at least as large" as $\overline{G}$. We call $\overline{G}$ the **closure** of G, also denoted *cl* G.

The following definition summarizes Examples 5 and 6:

Definition: Let G be a subset of (X, d). The **interior** of G, denoted *int* G or $\mathring{G}$, is the union of all open subsets of G, and the **closure** of G, denoted *cl* G or $\overline{G}$, is the intersection of all closed sets that contain G.

Remark: $\mathring{G} \subseteq G \subseteq \overline{G}$ — *i.e.*, *int* $G \subseteq G \subseteq$ *cl* G.

The interior of the complement of G — *i.e.*, *int* G^c — is called the *exterior* of G:

Definition: Let G be a subset of (X, d). The **exterior** of G, denoted *ext* G, is the interior of G^c.

Remark: The exterior of G is the union of all open sets that do not intersect G — *i.e.*, the largest open set in G^c.

The *boundary* of a set lies "between" its interior and exterior:

Definition: Let G be a subset of (X, d). The **boundary** of G, denoted $bdy\ G$, is the complement of $int\ G \cup ext\ G$ — *i.e.*, $bdy\ G = [int\ G \cup ext\ G]^c$.

Remark: The interior, exterior, and boundary of a set comprise a partition of the set. The interior and exterior are both open, and the boundary is closed. $bdy\ G = cl\ G \cap cl\ G^c$.

Example 7: Let $u : \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $u(x_1, x_2) = x_1 x_2$, and let $S = \{\mathbf{x} \in \mathbb{R}^2_{++} \mid u(\mathbf{x}) < \xi\}$ for some $\xi \in \mathbb{R}_{++}$. Note that $S = u^{-1}((-\infty, \xi))$, the inverse image under u of the open interval $(-\infty, \xi)$ — in fact, S is the inverse image of the open interval (c, ξ) for any $c \leqq 0$, and S is also the lower-contour set of any $\overline{\mathbf{x}}$ such that $u(\overline{\mathbf{x}}) = \xi$. We will show that S is an open set.

Proof: Let $\overline{\mathbf{x}} \in S$, *i.e.*, $\overline{x}_1 \overline{x}_2 < \xi$. We must find an $\epsilon > 0$ such that $B(\overline{\mathbf{x}}, \epsilon) \subseteq S$, *i.e.*, such that

$$\mathbf{x} \in B(\overline{\mathbf{x}}, \epsilon) \Rightarrow x_1 x_2 < \xi. \tag{2}$$

The arithmetic will be easiest if we use the max-norm $\|\cdot\|_\infty$, in which case, for whatever value of ϵ we choose, we will have

$$\begin{aligned} \mathbf{x} \in B(\overline{\mathbf{x}}, \epsilon) \quad &\Rightarrow \quad [x_1 < \bar{x}_1 + \epsilon \quad \& \quad x_2 < \bar{x}_2 + \epsilon] \\ &\Rightarrow \quad x_1 x_2 < (\overline{x}_1 + \epsilon)(\overline{x}_2 + \epsilon) \\ &\Leftrightarrow \quad x_1 x_2 < \overline{x}_1 \overline{x}_2 + (\overline{x}_1 + \overline{x}_2)\epsilon + \epsilon^2. \end{aligned}$$

Therefore we will have the desired implication (2) if $\overline{x}_1 \overline{x}_2 + (\overline{x}_1 + \overline{x}_2)\epsilon + \epsilon^2 = \xi$. We therefore solve the equation

$$\epsilon^2 + (\bar{x}_1 + \bar{x}_2)\epsilon + (\bar{x}_1 \bar{x}_2 - \xi) = 0$$

to obtain the ϵ we want (for the given quantities $\xi, \overline{x}_1$, and $\overline{x}_2$). The quadratic formula yields

$$\epsilon = \frac{1}{2}\left[-(\bar{x}_1 + \bar{x}_2) + \sqrt{(\bar{x}_1 + \bar{x}_2)^2 + 4(\xi - \bar{x}_1 \bar{x}_2)}\right] > 0,$$

and we have the desired implication (2). ■

Example 7 is an important example — not the specific utility function $u(x_1, x_2) = x_1 x_2$, but the fact that the set S we defined (the inverse image under u of an open set in $\mathbb{R}$) was itself an open set in the domain of u. In fact, it's a theorem we'll do later that a function $f : (X, d) \to (Y, d')$ is continuous if and only if $f^{-1}(A)$ is open in X for every open set $A \subseteq Y$. We've also said that we would define a preorder to be continuous if the strict upper- and lower-contour sets of every alternative are open sets. These notions of continuity, and their relationship to open sets, have important implications for utility theory and other applications in economics.

But suppose we change the example so that $u : \mathbb{R}^2_+ \to \mathbb{R}$ — *i.e.*, we've changed the domain of u from $\mathbb{R}^2_{++}$ to $\mathbb{R}^2_+$. Now the set $u^{-1}((-\infty, \xi))$ includes the boundary of the set $\mathbb{R}^2_+$, *i.e.*, the axes of $\mathbb{R}^2_+$. The inverse image of the set $(-\infty, \xi)$ does not appear to be an open set, as it was before. But the function $u(x_1, x_2) = x_1 x_2$ doesn't seem any less continuous than it was before — and indeed, when we define continuity, as we're going to do shortly, we'll find that u is continuous, even with the domain $\mathbb{R}^2_+$.

The key to this puzzle is in the condition, two paragraphs above, that for every open set A in the target space Y, the set $f^{-1}(A)$ must be "open in X" — *i.e.*, open *in the domain of* f. In Example 7, the domain of u is $\mathbb{R}^2_{++}$; in our variation on the example, the domain of u is $\mathbb{R}^2_+$. Each of these sets is a metric space, with the metric it inherits from $\mathbb{R}^2$, but they are not the *same* metric space as $\mathbb{R}^2$. Therefore a set might be open in one of these three metric spaces but not in another.

For example, let $\overline{\mathbf{x}} = (0, 2) \in \mathbb{R}^2$ and let $r = 1$, and consider how the open ball $B(\overline{\mathbf{x}}, r)$ changes as we move from the metric space $\mathbb{R}^2$ to the metric space $\mathbb{R}^2_+$ to the metric space $\mathbb{R}^2_{++}$. Recall the definition of the open ball $B(\overline{\mathbf{x}}, r)$:

Definition: Let (X, d) be a metric space, let $\overline{x} \in X$, and let r be a positive real number. The open ball about $\overline{x}$ of radius r is the set $B(\overline{x}, r) := \{x \in X \mid d(x, \overline{x}) < r\}$.

As the metric space playing the role of the definition's (X, d) changes from $X = \mathbb{R}^2$ to $X = \mathbb{R}^2_+$ to $X = \mathbb{R}^2_{++}$, the set $B(\overline{\mathbf{x}}, r)$ changes from, first, the whole disc about the point $(0, 2)$ of radius $r = 1$, to the "half-disc" which is the intersection of the whole disc and $\mathbb{R}^2_+$, to a set that's not defined — the set is not defined in the third case because $\overline{\mathbf{x}} \notin X$. But undefined sets like this did not appear in our proof in Example 7, because all the ϵ-balls we used were centered at points $\overline{\mathbf{x}} \in \mathbb{R}^2_{++}$.

In order to take a general approach to this issue, let's suppose we begin with a metric space (X, d), but then we want to work in a subspace (X', d'), where X' is a subset of X and d' is the metric that X' inherits from (X, d). For any open ball $B(\overline{x}, r)$ in X, let $B'(\overline{x}, r)$ be the corresponding open ball in X' — *i.e.*, $B'(\overline{x}, r) := B(\overline{x}, r) \cap X'$.

In our example we started with the metric space $X = \mathbb{R}^2$ and then we moved to the domain of u, which was the subspace $X' = \mathbb{R}^2_{++} \subseteq X$. Everything worked OK here because the set X' was an open subset of X, so for any point $\overline{\mathbf{x}} \in X'$, there is an open ball $B(\overline{x}, r)$ about $\overline{\mathbf{x}}$ that lies entirely in X' — therefore $B'(\overline{x}, r) = B(\overline{x}, r)$. Any subset $S \subseteq X'$ is open in the metric space X' if and only if it's open in the metric space X.

But when we moved instead to the subspace $X' = \mathbb{R}^2_+ \subseteq X$, that set was not an open set in the original metric space $X = \mathbb{R}^2$. Therefore there were sets, like the half-disc $B'(\overline{\mathbf{x}}, r)$ — *i.e.*, $B(\overline{\mathbf{x}}, r) \cap \mathbb{R}^2_+$ — that are open in the metric space X' but not open in the original metric space X.

This is a common occurrence. A useful concept to clarify it is the idea that a subset S of a subspace X' of X can be open *relative* to X':

Definition: Let (X, d) be a metric space, let $X' \subseteq X$, and let $S \subseteq X'$. The set S is **open relative to** X' if S is an open set in the metric space X', with the metric it inherits from (X, d). We also say simply that S is open in X'.

The following remark provides a useful characterization of relatively open sets.

Remark: If S is a subset of $X' \subseteq X$, then S is open relative to X' if and only if there is an open set V in X such that $S = V \cap X'$.

Proof:

($\Rightarrow$): Suppose S is open relative to X'. Then for every $x \in S$ there is an $\epsilon_x > 0$ such that $B'(x, \epsilon_x) \subseteq S$. But also $B(x, \epsilon_x)$ is open in X. Let $V = \cup_{x\in S} B(x, \epsilon_x)$; then V is open in X and $S = V \cap X'$, as follows:

$$\cup_{x\in S} B'(x, \epsilon_x) = S$$

$$i.e.,\ \cup_{x\in S} \left[B(x, \epsilon_x) \cap X'\right] = S$$

$$i.e.,\ \left[\cup_{x\in S} B(x, \epsilon_x)\right] \cap X' = S$$

$$i.e.,\ V \cap X' = S.$$

($\Leftarrow$): Suppose V is an open set in X such that $S = V \cap X'$. We must show that S is open relative to X'. Let $x \in S$ and let $\epsilon > 0$ be such that $B(x, \epsilon) \subseteq V$ [which we can do because V is open in X]. Then also $B'(x, \epsilon) \subseteq V$, and of course $B'(x, \epsilon) \subseteq X'$. Therefore $B'(x, \epsilon) \subseteq V \cap X'$ — *i.e.*, $B'(x, \epsilon) \subseteq S$. Since we chose $x \in S$ arbitrarily, this establishes that S is open relative to X'. ■

Of course, we can do exactly the same thing for closed sets:

Definition: Let (X, d) be a metric space, let $X' \subseteq X$, and let $S \subseteq X'$. The set S is **closed relative to** X' if S is a closed set in the metric space X', with the metric it inherits from (X, d). We also say simply that S is closed in X'.

And we have the parallel characterization of relatively closed sets:

Remark: If S is a subset of $X' \subseteq X$, then S is closed relative to X' if and only if there is a closed set G in X such that $S = G \cap X'$.

Proof: Exercise.

Let's return to Example 7, but let's define u on the domain $\mathbb{R}^2_+$ — *i.e.*, $u : \mathbb{R}^2_+ \to \mathbb{R}$ instead of $u : \mathbb{R}^2_{++} \to \mathbb{R}$. Although the lower-contour set $S = \{x \in \mathbb{R}^2_+ \,|\, u(\mathbf{x}) < \xi\}$ is not an open set in $\mathbb{R}^2$ (that's the issue that's led us to the notion of *relatively* open sets), S *is* open in $\mathbb{R}^2_+$, the domain of u — *i.e.*, S is open *relative to* the domain of u — because S is the intersection of $\mathbb{R}^2_+$ (the domain of u) and the set $\{x \in \mathbb{R}^2 \,|\, u(\mathbf{x}) < \xi\}$, which is open in $\mathbb{R}^2$. So this is consistent after all with the function u and the associated preference preorder $\succsim$ being continuous on the consumption set $\mathbb{R}^2_+$.