A simple incentive compatible scheme for attaining Lindahl allocations

By Mark Walker

A simple scheme for making governmental decisions about the production and financing of public goods is presented. The "competitive" equilibria under the scheme are Pareto optimal; more importantly, they are Lindahl equilibria. Thus, it is never in any individual's interest to refuse to participate (no one will be worse off at the equilibrium than at his initial holding); moreover, the existence of equilibria is assured in the usual classical public-goods economies.

Until recently, it was believed that any attempt to determine Pareto optimal allocations in the presence of public goods would be thwarted by individuals' incentives to misrepresent their preferences—to be "free riders," as it were. Groves and Ledyard [2] upset this belief when they introduced a method for determining the production and financing of public goods at Pareto optimal levels, thus apparently overcoming the free rider problem. Their scheme consists of a government which solicits information from consumers concerning their preferences for the various public goods, and which uses the information it receives to determine (according to specified rules) both its purchases of public goods and the taxes that will be levied on consumers to finance those purchases. When this governmental "mechanism" is employed in conjunction with competitive markets (for allocating private goods), and when consumers follow "competitive-like" Cournot behavior in choosing the information they will convey to the government, the resulting equilibria are Pareto optimal (it should be noted, however, that sometimes there is no equilibrium; see [3]).

An alternative to the Groves-Ledyard mechanism will be presented here. The mechanism is similar in many respects to the one they have introduced; everything that I have said, for example, in the preceding paragraph describing the Groves-Ledyard mechanism will be true of this new mechanism as well. The new mechanism has several advantages, however. First, it is remarkably simple; indeed, it is transparent how and why the scheme works as it does. Second, and most important, the equilibria of this new mechanism coincide precisely with the Lindahl equilibria: Lindahl prices are generated and Lindahl allocations are attained, in spite of the fact that individuals behave strategically and follow only their own self-interest.

The achievement of Lindahl outcomes has two important consequences. First, it means that no individual will ever be worse off after participating in the mechanism (that is, no one will ever be worse off at one of the mechanism's equilibria) than he was when he had only his initial holding of goods. This feature is in sharp contrast with the Groves-Ledyard mechanism, in which some individuals will generally find it in their interest not to participate, because the

1 The results contained in this paper were first reported in "An Auctioneerless Mechanism for Attaining Lindahl Allocations," 1977. I would like to thank Thomas Muench and Robert Willis for helpful comments.
mechanism will leave them worse off than in their original situation. Second, it is known that Lindahl equilibria exist in almost exactly those public-goods economies which would admit competitive equilibria if all public goods were private goods instead, and the proof of such existence is essentially the same as proving the existence of competitive equilibria when there are only private goods (contrast this with the complexity of the existence question for the Groves-Ledyard mechanism [2, 3]).

Because the mechanism that I will describe achieves Lindahl allocations and Lindahl prices at its equilibria if economic agents practice a combination of competitive and Cournot behavior, I will refer to the mechanism as the Cournot-Lindahl mechanism. If one is not careful, this terminology could be misleading in at least two respects: it is by no means the only mechanism that attains Lindahl allocations at its Cournot equilibria\(^2\) (although it is probably the simplest and most transparent one); and the word “Cournot” actually refers to the participants’ posited behavior (specifically, to the solution concept we will consider), not to the mechanism per se. In spite of these caveats, however, the Cournot-Lindahl terminology is really quite a good description of the mechanism’s performance.

In order to give the clearest possible exposition of the Cournot-Lindahl mechanism, I will first define it for the simplest of economies, those in which there are only two goods (one public and one private) and a constant-returns technology for producing the public good from the private one. Quantities of the private good will be denoted by \(x\), and quantities of the public good by \(y\). There are \(n\) consumers, indexed \(i = 1, \ldots, n\), and each characterized by (i) an endowment \(\omega_i > 0\) of the private good, and (ii) a utility function \(u_i : X_i \rightarrow R\), where the domain \(X_i\) of \(u_i\), called \(i\)’s consumption set, is a subset of \(R^2_+\) which includes the pair \((\omega_i, 0)\). An allocation is an \((n + 1)\)-tuple \((y, x_1, \ldots, x_n) \in R_+ \times R^*_n\); \(y\) will be interpreted as the level at which the public good is provided, and \(x_i\) as the quantity of private good allocated to consumer \(i\), whose preference over the allocations is defined by applying \(u_i\) to \((x_i, y)\).

The economy has no endowment of the public good, but can produce it by using the private good as input in a linear production process. Each unit of the public good requires \(\beta\) units of input \((\beta > 0)\), so we have the following feasibility constraint:

\[
\sum_{i=1}^{n} x_i + \beta y \leq \sum_{i=1}^{n} \omega_i.
\]

In keeping with the general framework suggested by Groves and Ledyard, the mechanism by which allocations are determined (in the simple two-good economy) will be a government. The government solicits information from the

\(^2\) In [4], Hurwicz devised a mechanism in which the Cournot-Nash equilibria coincide with the Lindahl equilibria. However, the equilibria in that mechanism are extremely unsatisfactory in other respects: they are very unstable (at any disequilibrium, many individuals’ maximizing choices remain very far from their equilibrium choices); and at any equilibrium many individuals are indifferent between their equilibrium choice and a large class of other (non-equilibrium) choices. These stability and strength-of-incentive weaknesses are not related to the size of the economy, as in Muench and Walker [9, 10]; they are present in Hurwicz’s mechanism for even as few as three individuals.
players, and then uses the information it receives to determine both the quantity of the public good it will purchase and the tax obligations of each of the \( n \) players. Thus, a government is defined by specifying

\begin{enumerate}
\item for each player \( i \in N \): the set \( M_i \) of messages that player \( i \) is allowed to report; and
\item the \( n+1 \) outcome functions \( \Psi: \prod_i^n M_i \to \mathbb{R}_+ \) and \( \tau_i: \prod_i^n M_i \to \mathbb{R} \) (\( i = 1, \ldots, n \)) which specify, respectively, the public good level \( y \) and the taxes \( t_1, \ldots, t_n \), all as functions of the \( n \)-tuple \( m = (m_1, \ldots, m_n) \) of messages. (In the simple two-good economy, we will take the tax levels \( t_i \) to be simply amounts of the private good, with negative tax levels permitted.)
\end{enumerate}

Notice that a government defines, in a natural way, an \( n \)-person game: each player’s strategy space is his set \( M_i \) of messages; the outcomes are the \( (n+1) \)-tuples \( (y, t_1, \ldots, t_n) \in \mathbb{R}_+ \times \mathbb{R}^n \); the outcome function is the function \( (\Psi, \tau_1, \ldots, \tau_n): \prod_i^n M_i \to \mathbb{R}_+ \times \mathbb{R}^n \); and player \( i \)’s payoff function is given by applying his utility function \( u_i \) to the pair \((\omega_i - t_i, y)\).

Let us define the Cournot-Lindahl government as follows:

\begin{enumerate}
\item \( M_i = \mathbb{R} \) \quad (i = 1, \ldots, n);
\item \( \Psi(m) = \sum_{i=1}^n m_i \);
\item \( \tau_i(m) = \frac{1}{n} \beta + m_{i+2} - m_{i+1} \Psi(m), \quad i = 1, \ldots, n, \) where the subscripts \( n + 1 \) and \( n + 2 \) are understood as “modulo \( n \)” — i.e., \( n + 1 = 1 \) and \( n + 2 = 2 \).
\end{enumerate}

The Cournot-Nash equilibria of the game defined by this government correspond exactly to the economy’s Lindahl equilibria. This correspondence between the two kinds of equilibrium is easy to see; it can be demonstrated in just a few sentences, as follows.

Suppose that each player behaves according to the Cournot-Nash assumption, each treating the other \( n - 1 \) players’ messages as parametric when choosing his own message. Then each player’s decision problem is precisely the one suggested by Erik Lindahl in [7] (the suggestion is much clearer in L. Johansen [6]). Specifically, player \( i \) can, by his choice of \( m_i \), choose (i.e., “demand”) any public good level \( y \) (that he can afford), but for each unit of the public good so chosen he must pay a price \( q_i = (1/n) \beta + m_{i+2} - m_{i+1} \). Furthermore, since the player is taking the others’ messages as given, he is in particular taking his own personal price \( q_i \) as given. Since \( \sum_{i=1}^n q_i \) is identically \( \beta \), it is obvious that any equilibrium \( n \)-tuple

\textsuperscript{3} The game is not completely well-defined, a weakness that the C-L mechanism shares with the Groves-Ledyard mechanism: there are some message \( n \)-tuples \( m \) which would leave some players outside their consumption sets \( X_i \). Note, however, that each interior equilibrium has a neighborhood on which feasibility is assured. Hurwicz, Maskin, and Postlewaite [5] have treated this problem in some detail for economies with only private goods.
\( m = (m_1, \ldots, m_n) \) will correspond to a Lindahl equilibrium: the resulting allocation will be a Lindahl allocation, and the prices \( q_1, \ldots, q_n \) will be a Lindahl price configuration.

It is almost as easy to see the converse, that each Lindahl equilibrium must correspond to a Cournot-Nash equilibrium of the game defined by the C-L government. Suppose that \( (q_1, \ldots, q_n) \) is a Lindahl price list and that \( (y, x_1, \ldots, x_n) \) is the corresponding Lindahl allocation. Consider the following system of \( n \) linear equations in the \( n \) variables \( m_1, \ldots, m_n \):

\[
\begin{align*}
m_1 + m_2 + \ldots + m_n &= y, \\
m_2 - m_1 &= q_n - \frac{1}{n} \beta, \\
m_3 - m_2 &= q_1 - \frac{1}{n} \beta, \\
\vdots & \quad \vdots \\
m_n - m_{n-1} &= q_{n-2} - \frac{1}{n} \beta.
\end{align*}
\]

It is easy to verify that the determinant of the \( n \times n \) coefficient matrix of this system has value \( n \); thus, the system can be solved for the \( n \)-tuple \( (m_1, \ldots, m_n) \), which is clearly a Cournot-Nash equilibrium.

It is straightforward to generalize the C-L government to arbitrary numbers of goods and to more general production relations. Suppose that there are \( L \) private goods, denoted \( l = 1, \ldots, L \), with market prices denoted \( p_l \); \( K \) public goods, denoted \( k = 1, \ldots, K \), with market prices denoted \( q_k \); \( J \) production sets \( Y_j \subseteq R^L \times R^K \), \( j = 1, \ldots, J \), with elements \( (z_j, y_j) \); \( n \) consumers, each with an endowment \( \omega_i \in R^L \) of private goods, a consumption set \( X_i \subseteq R^L + R^K \) (with \( (\omega_i, 0) \in X_i \)), and a utility function \( u_i : X_i \rightarrow R^+ \); and \( nJ \) ownership shares \( \theta_{ij} \in R_+ \), satisfying \( \sum_{j=1}^n \theta_{ij} = 1, j = 1, \ldots, J \). A government is defined as before, except that now the public-goods outcome function \( \mathcal{Y} \) must have \( K \) components, \( \mathcal{Y}_k : \Pi^n M_i \rightarrow R_+ \) \( (i = 1, \ldots, K) \).

The C-L government is now defined by

\[
\begin{align*}
M_i &= R^K & (i = 1, \ldots, n); \\
\mathcal{Y}_k(m) &= \sum_{i=1}^n m_{ik} & (k = 1, \ldots, K); \\
\tau_i(m) &= \sum_{k=1}^K q_{ik}(m) \mathcal{Y}_k(m) & (i = 1, \ldots, n),
\end{align*}
\]

where

\[
q_{ik}(m) = \frac{1}{n} q_k + m_{i+2,k} - m_{i+1,k} \quad \text{for each } i \text{ and } k.
\]

The tax \( \tau_i(m) \) to be paid by consumer \( i \) is now interpreted as an amount of "money" (i.e., units of account).
As in [2], the equilibria we will investigate are those arising from competitive behavior in production decisions and private-goods purchasing decisions, and Cournot-Nash behavior in the information-reporting decisions of consumers. Formally, a C-L equilibrium is a list
\[(p^*, q^*), (x_i^*, m_i^*)_{i=1}^n, (z_j^*, y_j^*)_{j=1}^J\]
satisfying the conditions
\[(E.1) \sum_{i=1}^n x_{il}^* = \sum_{j=1}^J z_{jl}^* + \sum_{i=1}^n \omega_{il} \quad (l = 1, \ldots, L);\]
\[(E.2) \sum_{i=1}^n m_i^* = \sum_{j=1}^J y_j^*;\]
\[(E.3) (z_j^*, y_j^*) \in Y_j \quad (j = 1, \ldots, J);\]
\[(E.4) (z_j^*, y_j^*) \text{ maximizes } p^* \cdot z_j + q^* \cdot y_j \text{ over } Y_j, \quad j = 1, \ldots, J;\]
\[(E.5) (x_i^*, \sum_{j=1}^J y_j^*) \in X_i \quad (i = 1, \ldots, n);\]
\[(E.6) (x_i^*, y_i^*) \text{ maximizes } u_i \text{ among all the } (x_i, y) \in X_i \text{ for which there is some } m_i \in R^k \text{ satisfying}\]
\[m_i + \sum_{l \neq i} m_l^* = y\]
and\(^4\)
\[\sum_{l=1}^L p_i^* x_{il} \leq \sum_{l=1}^L p_i^* \omega_{il} - \tau_i(m_i, m_i^*) + \sum_{j=1}^J \theta_{ij}(p^* \cdot z_j^* + q^* \cdot y_j^*).\]

The arguments that were used in the two-good case can be used to establish the same result here: that the C-L equilibria correspond exactly to the Lindahl equilibria. On the one hand, each consumer’s decision problem is still identical to his Lindahl decision problem, so that each C-L equilibrium yields a Lindahl equilibrium. And on the other hand, for any Lindahl equilibrium
\[((\hat{p}, \hat{q}), (\hat{q}_{i,k})_{i=1}^n, (\hat{x}_i)_{i=1}^n, (\hat{z}_j, \hat{y}_j)_{j=1}^J),\]
the system of equations (6) can be replaced, for each public good \(k\), by the system
\[m_{1k} + \ldots + m_{nk} = \hat{y}_k,\]
\[m_{zk} - m_{1k} = \hat{q}_{nk} - \frac{1}{n} \hat{q}_k,\]
\[\vdots \]
\[m_{nk} - m_{n-1,k} = \hat{q}_{n-2,k} - \frac{1}{n} \hat{q}_k,\]

\(^4\)The notation \((m_i, m_i^*)\) represents the \(n\)-tuple \((m_1^*, \ldots, m_i^*, m_i, m_i^*, \ldots, m_n^*)\).
and solved as before for $m_{1k}, \ldots, m_{nk}$. This establishes that for each Lindahl equilibrium, there is a corresponding C-L equilibrium. We thus have the following theorem.

**Theorem:** In an economy with three or more consumers there is a one-to-one correspondence between the set of C-L equilibria and the set of Lindahl equilibria; in the equilibria which correspond to one another the allocations are identical and the Lindahl prices $q_{ik}$ and the individuals' C-L messages $m_{ik}$ are related via

$$q_{ik} = \frac{1}{n} m_{i+2,k} - m_{i+1,k} \quad (i = 1, \ldots, n \text{ and } k = 1, \ldots, K)$$

and

$$q_k = \sum_{i=1}^{n} q_{ik} \quad (k = 1, \ldots, K).$$

As a corollary, we know that C-L equilibria exist in precisely those economies that have Lindahl equilibria. (For sets of conditions that are sufficient to insure that an economy has a Lindahl equilibrium, the reader is referred to D. Foley [1] or J.-C. Milleron [8]). And as a second corollary, we have the two fundamental theorems of welfare economics (when the economy satisfies the usual conditions for these theorems): each C-L equilibrium is Pareto optimal, and each Pareto optimum can be achieved as a C-L equilibrium, with appropriate redistribution of initial holdings.

A Cournot-Lindahl government can be defined in an analogous way for economies in which there is a continuum of individuals.\(^5\) The following version, described in the simple two-goods constant-returns economy, also makes clear that the Cournot-Lindahl government need not base an individual’s Lindahl price upon the messages of only two other individuals.

Let the real unit interval $A = [0, 1]$ be the set of all the individuals; an allocation is a pair $(x, y)$, where $y \in R_+$ and $x$ is an integrable real function on $A$. Let $b$ be the per-capita unit cost of the public good; i.e., the feasible allocations are the ones that satisfy the inequality

$$\int_A x + by \leq \int_A \omega.$$

For each individual $a \in A$, define the sets $R(a)$ and $L(a)$ as follows: if $a = \frac{1}{2}$, then $L(a) = [0, a)$ and $R(a) = (a, 1]$; if $a < \frac{1}{2}$, then $L(a) = [0, a) \cup [a + \frac{1}{2}, 1]$ and $R(a) = \left(a, a + \frac{1}{2}\right)$; if $a > \frac{1}{2}$, then $L(a) = \left[a - \frac{1}{2}, a\right)$ and $R(a) = (a, 1] \cup [0, a - \frac{1}{2})$. A message complex for the Cournot-Lindahl government is an integrable real function $m$ on $A$; let the outcome functions $y = \vartheta(m)$ and $t_a = \tau_a(m)$ be given by

$$y = \int_A m \quad \text{and} \quad t_a = b + \int_{R(a)} m - \int_{L(a)} m.$$
It is easy to see that the Cournot-Lindahl equilibria and the Lindahl equilibria again coincide with one another, just as in the finite case, although here each individual's Lindahl price depends upon all other individuals' messages (the same thing can clearly be done in the finite case, as well).

In summary, then, it is really quite easy to design mechanisms—apparently very "nice" mechanisms—whose Cournot-Nash equilibria yield Lindahl allocations and Lindahl prices. This fact suggests two quite different conclusions. First, it suggests that we should not dismiss out of hand, simply on the ground that it is "incompatible with individual incentives," Lindahl's approach to the financing of public goods. Of course, it might nevertheless be true that Lindahl allocations really are incompatible with individuals' actual behavior (see, e.g., [12]). In that case, the performance of the Cournot-Lindahl mechanism suggests that the question whether a potential outcome is incentive-compatible cannot be answered by simply determining whether that outcome is attainable as a Cournot-Nash equilibrium. More generally, if some mechanisms or some outcomes are really "more compatible" than others with individuals' actual behavior, and if we want to develop a theory that will help us to understand and explain that fact, then we will have to do more than simply appeal to one or another equilibrium concept. The issue of incentive-compatibility seems to be a good deal more subtle than that, and will apparently require a deeper analysis than we have yet developed.

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