ON THE NONEXISTENCE OF A DOMINANT STRATEGY MECHANISM FOR MAKING OPTIMAL PUBLIC DECISIONS

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In a broad class of situations not covered by the Gibbard–Satterthwaite Theorem it is shown that one cannot design a strategy-proof choice mechanism which attains Pareto optimal outcomes. The results are shown to be generic in character—i.e., any non-manipulable mechanism will attain nonoptimal outcomes virtually everywhere—and they cover, in particular, certain problems of allocating public and private goods. The analysis is carried out in transferable utility environments, and makes extensive use of the mechanisms recently introduced by Groves.

GIBBARD [1] AND SATTERTHWAITE [17] have recently shown that one cannot design satisfactory institutions (or “social decision mechanisms”) that are immune to strategic manipulation. They demonstrated that under any reasonable mechanism, a participant will sometimes, by acting as if his preference were other than the one that he really holds, be able to secure an outcome that he will prefer over the outcome had he acted sincerely. Perhaps the most familiar example of this phenomenon occurs in rank-order voting, where a voter will sometimes assign to his second-most-preferred alternative the lowest ranking on his ballot, in order to keep that alternative from defeating the one that he most prefers.

There are many important social decision problems which fall outside the purview of the Gibbard–Satterthwaite Theorem. The theorem applies only to those problems in which there is a finite number of alternative social states, and in which no individual preference over those states can be ruled from consideration a priori. This excludes, for example, many problems of resource allocation, in which it is often useful to represent the set of alternative states as a Euclidean space, or to ask only that a mechanism make choices when individual preferences are relatively “nice:” when individuals care only about certain components of the social state (only one’s own allocation, for example); when preferences are continuous; perhaps only when preferences are convex and/or monotone. Since they are not covered by the Gibbard–Satterthwaite Theorem, we might hope for a more encouraging result in such cases. Indeed, Groves [5] has discovered that for one important class of social decision problems—essentially those in which utility is transferable—there is a class of mechanisms which are immune to manipulation: each participant in a “Groves mechanism” always finds that behaving sincerely (or truthfully) is a dominant strategy (that is, sincere behavior is best for him, no matter which strategies are chosen by the other participants.)

My objective in this article is to show that the dominant strategy property can generally be obtained only by sacrificing Pareto optimality of some of the outcomes. That result will be established (as Theorem 4) whenever the admissible environments include a (sufficiently diverse) class in which utility is transferable. The result is of added interest because it includes the standard public goods model (in the case where there is one private good, in terms of which utility can be

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1 Several discussions with Thomas Muench were most helpful.

1521
measured), and it therefore demonstrates that such public goods economies are inherently manipulable. One cannot devise a mechanism, in other words, that will always provide for Pareto optimal allocation and financing of public goods, and that will at the same time be immune to strategic manipulation.

Further, this impossibility result is shown to have the following generic character. If we ask not merely for a dominant strategy mechanism, but for a mechanism in which sincere behavior is always dominant (i.e., a mechanism in which no one will ever misrepresent himself), then the only mechanisms available to us are ones which produce nonoptimal outcomes not merely in some environments, but in a very large class of environments (viz., a class that is dense, in a well defined sense). If we demand a mechanism that is continuous, as well, then we will virtually always have nonoptimal outcomes (i.e., such outcomes will occur on an open dense set of environments).

To the best of my knowledge, the generic results that I will present here are the first of their kind. Related nonexistence results, however, have been obtained by others. In [9], for example, Hurwicz demonstrated that for classical private goods economies, no mechanism can completely eliminate strategic behavior if it is also required to produce Pareto optimal outcomes and to leave each participant with an allocation that he likes at least as well as his initial one (the latter property is sometimes called "individual rationality")." Ledyard and Roberts [12] have verified that Hurwicz's result is unaffected by the presence of public goods. It can be shown, using Theorem 3, below, that both of these results are generic, in the sense already described, in transferable utility economies.

The individual rationality property used in the preceding paragraph is not so compelling when there are public goods as it is when there are only private goods. The nonexistence results in the present article make no use of that property. Similar nonexistence results have been obtained independently by Hurwicz [10] and by Green and Laffont [4]. The Hurwicz result covers the case in which there are exactly three players and in which the mechanisms must be quite smooth (thrice differentiable); the Green–Laffont result covers the case in which utility is transferable, but in which the individual preferences are otherwise unrestricted (the result is therefore most naturally applicable to cases in which the set of public alternatives has no linear structure). For purposes of comparison, the results that I will present here apply to arbitrary mechanisms, and to arbitrary continuous mechanisms; to any number of players; and to environments in which the preferences are convex. And they are shown to hold virtually everywhere.

The results will be established by first concentrating upon Groves mechanisms. It will be shown in Section 3 (in Theorems 1 and 2) that every Groves-mechanism yields nonoptimal outcomes on a class of environments that is dense (open and dense, if the mechanism is continuous) in the class of all concave environments. The remaining results, in Section 4, follow straightforwardly from Theorems 1 and 2 via a theorem of Walker [18], which (following Green and Laffont [3]) characterizes all truth-dominant mechanisms as Groves mechanisms. The first two theorems, in turn, follow from a fundamental property of Groves-type mechanisms, a property that is established in the Cubical Array Lemma in Section
2. Section 1 introduces the formal structure of the problems to be analyzed, in a series of definitions and elementary remarks. Section 5 considers applications to certain public and private good allocation problems, and Section 6 contains some remarks concerning the broader implications of the impossibility results.

1. MECHANISMS AND THEIR PROPERTIES

Suppose that we have \( n \) players, each of whom we will identify with one of the first \( n \) positive integers; let \( N \) denote the set \( \{1, \ldots, n\} \) of all the players. A social state will be described by \( n + 1 \) components: for \( i = 1, \ldots, n \), component \( i \) will be the component about which only player \( i \) cares; the \((n + 1)\)st component will be one about which all the players generally care. More formally, a state is an \((n + 1)\)-tuple \((x, y) = (x_1, \ldots, x_n, y) \in (\prod_{i=1}^{n} X_i) \times Y = X \times Y\), and each player is assumed to have a utility function \( u_i : X \times Y \to \mathbb{R} \) which depends only upon "his own" component \( x_i \) and the "public" component \( y \) of the social state.

The analysis will be carried out in a transferable utility framework. Thus, we let \( X_i = \mathbb{R} \) for each \( i \in N \), and we consider only those utility functions which have the form

\[
u_i(x, y) = x_i + v_i(y),\]

where \( v_i \) is a real-valued function on \( Y \), called player \( i \)'s valuation. A profile (over the set \( Y \)) is an \( n \)-tuple \( v = (v_1, \ldots, v_n) \) of valuations on \( Y \). When we need to be clear about the number of players involved, we will refer to a profile with \( n \) components as an \( n \) profile. We will always interpret the component \( x_i \) of a state as the increment of utility (or of the private good) accruing to player \( i \); thus, we define the feasible states to be the ones which satisfy the condition

\[
\sum_{i=1}^{n} x_i \leq 0.
\]

Suppose that we require each player to report his utility function \( u_i \) (or equivalently, his valuation \( v_i \)) to some public agency which is charged with choosing a social state \((x, y)\). We would like to specify a decision rule that the agency could follow which would allow it to transform reported profiles \( v = (v_1, \ldots, v_n) \) into outcomes \((x, y) = (x_1, \ldots, x_n, y)\). For each player \( i \in N \), let \( V_i \) be a set of admissible valuations \( v_i \); then a decision rule, or mechanism, for the public agency is simply a function \( m : \prod_i^n V_i \to \mathbb{R}^n \times Y \). We write

\[
m = (\tau, \pi) = (\tau_1, \ldots, \tau_n, \pi),
\]

where \( \tau_i : \prod_i^n V_i \to \mathbb{R} \) is called the transfer function for player \( i \), and \( \pi : \prod_i^n V_i \to Y \) is called the public decision function.

We will always treat the sets \( V_1, \ldots, V_n \) as given and fixed (and as if they are known to all players and to the public agency). We are interested in the "design problem" of choosing \( m \) according to some specified criteria. It is assumed that

\[\text{2 Lower bounds on the set of feasible states will be considered in Section 5.}\]
once a mechanism \( m \) is specified, the agency will adhere to the mechanism, and the players will know that the agency is using \( m \).

The restriction to mechanisms which require each player to report his utility function, or valuation, seems at first to be a severe limitation. It seems preferable, instead, to consider mechanisms which allow a player to transmit other (and especially simpler) information than an entire utility function. Let us follow Hurwicz [9], however, in requiring that each player behave (in whatever kind of mechanism he is faced with) in a manner that is at least consistent with some (admissible) utility function \( u_i \). Whether, in a particular case, a player has chosen to rationalize his behavior with his true utility function, or with some other one, is, of course, not discernable to an outside observer. However, any behavior that cannot be so rationalized can be detected and punished as a violation of the rules. Now as a player's participation in the mechanism begins, he must clearly choose a valuation \( v_i \in V_i \) on which to base his behavior. Whether or not the mechanism literally requires the players to report their valuations is now irrelevant: in any event, the mechanism must operate to transform the chosen valuations \( v = (v_1, \ldots, v_n) \) into an outcome, as we have formally specified in our function \( m \). It is simply a helpful device to imagine that this is accomplished by the players reporting their valuations to some agency.

1.1. **Dominant Strategy Mechanisms**

In general, the public agency cannot expect the players to report their true valuations: unless the mechanism is constructed with some care, a player will often be able to manipulate the outcome to his advantage by misrepresenting himself (i.e., by reporting some false valuation). In particular, it is not enough for the public agency to simply choose, for each reported profile \( v \in \Pi^n V_n \), an outcome \( m(v) \) which is Pareto optimal with respect to \( v \). If some players are misrepresenting themselves, then \( m(v) \) is not likely to be truly Pareto optimal—i.e., it is not likely to be Pareto optimal with respect to the true (but unobservable) profile, say \( \hat{v} \). The reported valuations, however, are all the information that the agency has to work with. The question, then, is whether it is possible to construct any mechanism \( m \) which will always induce the players to report truthfully and will still yield Pareto optimal outcomes \( m(v) \); or whether (and we shall see that this comes to the same thing) we can construct a mechanism in which, even if the players are misrepresenting themselves, \( m(v) \) is nevertheless always Pareto optimal with respect to the true profile \( \hat{v} \) which generated the report \( v \).

Given a mechanism and a profile of true valuations, we can describe the players' strategic considerations in game-theoretic terms. The strategies available to player \( i \) are always the members of \( V_i \): player \( i \) is free to report any one of his admissible valuations. The outcome function of the game is simply the mechanism \( m : \Pi^n V_n \rightarrow \mathbb{R}^n \times Y \) that is being used. And the profile of true valuations, say \( \hat{v} \), defines a payoff function \( \hat{u}_i(x, y) = x_i + \hat{v}_i(y) \) for each player. Thus, for any sets \( V_1, \ldots, V_n \), any mechanism \( m \), and any profile \( \hat{v} \), we have an \( n \)-person game in normal form.
A game of the sort we have just described is strategically trivial if each player has a dominant strategy—i.e., a strategy which maximizes his payoff no matter what strategies the other players choose. A mechanism will therefore be free of strategic considerations if it always provides each player with a dominant strategy (i.e., if it does so for each $\hat{v} \in \Pi^i_i V_i$). We formalize these ideas with the following notation and definitions.

**Notation:** For any subset $S$ of $N$, the notation $v_S$ will denote a list assigning to each member $i$ of $S$ a valuation in his admissible set $V_i$. The notation $v_{-i}$ will be used for the $(n-1)$-tuple $v_{-(i)}$. The notation $(v_S, v_{-S})$ will denote the $n$-tuple $v \in \Pi^i_i V_i$ in which the $i$th component is selected from $v_S$ if $i \in S$, and from $v_{-S}$ if $i \not\in S$.

**Definition 1:** A strategy $v_i \in V_i$ is dominant for $i$ (with respect to $m : \Pi^i_i V_i \to \mathbb{R}^n \times Y$ and $\hat{v}_i \in V_i$) if, for every $\hat{v} \in \Pi^i_i V_i$, $\hat{v}_i(m(\hat{v}_{-i}, v_i)) \equiv \hat{v}_i(m(\hat{v}))$, where $\hat{u}_i(x, y) = x_i + \hat{v}_i(y)$.

**Definition 2:** A mechanism $m = (\tau, \pi) : \Pi^i_i V_i \to \mathbb{R}^n \times Y$ is said to be a dominant strategy mechanism if it satisfies the following condition (called the dominance condition):

**Condition D:** For each $i \in N$ and each $v_i \in V_i$, there is a dominant strategy for $i$.

A special class of dominant strategy mechanisms consists of those for which each player’s true valuation is always dominant.

**Definition 3:** A mechanism $m : \Pi^i_i V_i \to \mathbb{R}^n \times Y$ is said to be truth dominant if it satisfies the following condition (called the truth dominance condition):

**Condition T:** For each $i \in N$ and each $\hat{v}_i \in V_i$, the strategy $\hat{v}_i$ is dominant for $i$.

1.2. *Pareto Efficient Mechanisms*

Of course, if our only desideratum in choosing a mechanism is that Condition D or T always be satisfied, then the design problem is trivial: any constant mechanism $m$ will do. The problem becomes considerably more interesting when we require as well that a mechanism must yield outcomes that are good ones, in some sense. Accordingly, we will ask that our mechanism always attain Pareto optimal outcomes, whenever such outcomes exist.

In our transferable utility world, the Pareto optimal states are precisely the ones which maximize the (true) aggregate utility, $\Sigma^i_i \hat{u}_i(x, y) = \Sigma^i_i x_i + \Sigma^i_i \hat{v}_i(y)$. Since the only feasible states are the ones which satisfy

\[
\sum_{i=1}^{n} x_i \leq 0,
\]
the Pareto optimal states can be characterized as the ones which, at the same time, maximize the aggregate valuation \( \Sigma^n_i v_i : Y \to \mathbb{R} \) and also balance the public agency’s budget:

\[
\sum_{i=1}^{n} x_i = 0.
\]

Let \( \varphi : Y \times \prod^n_i V_i \to \mathbb{R} \) be the function defined by

\[
\varphi(y, v) = \sum_{i=1}^{n} v_i(y).
\]

For each \( v \in \prod^n_i V_i \), the aggregate valuation is the function \( \varphi(\cdot, v) : Y \to \mathbb{R} \).

**Definition 4:** Let \( V_1, \ldots, V_n \) be sets of valuations on a set \( Y \). A function \( \pi : \prod^n_i V_i \to Y \) is valuation-maximizing at \( v \in \prod^n_i V_i \) if

\[
\pi(v) \text{ maximizes } \varphi(\cdot, v).
\]

An \( n \)-tuple \( \tau \) of functions \( \tau_i : \prod^n_i V_i \to \mathbb{R} \) \( (j = 1, \ldots, n) \) is budget balancing at \( v \) if

\[
\sum_{j=1}^{n} \tau_j(v) = 0.
\]

A mechanism \( m = (\tau, \pi) : \prod^n_i V_i \to Y \times \mathbb{R}^n \times Y \) is budget balancing (resp.: valuation maximizing) at \( v \) if \( \tau \) is budget balancing at \( v \) (resp.: \( \pi \) is valuation maximizing at \( v \)).

Of course, a function \( \pi \) could not be valuation maximizing at a profile \( v \) if the aggregate valuation were to have no maximum—i.e., if there were no Pareto optimal state for the profile \( v \). We want to say that a mechanism is Pareto efficient if it attains Pareto optimal outcomes whenever such outcomes do exist. If we formalize that definition, we will have to have a notation for the subset of \( \prod^n_i V_i \) on which Pareto optima exist. That subset will not itself be a product (in general), and that will cause a considerable amount of unpleasantness in the statements and proofs of the theorems. The analysis will proceed much more smoothly if we sacrifice a bit of generality, and require at the outset that each admissible valuation attain a maximum—i.e., that each set \( V_i \) consist only of maximum attaining valuations. Then every profile \( v \) will admit Pareto optimal states, and we can make the following definition.

**Definition 5:** For each \( i \in N \), let \( V_i \) be a set of maximum attaining valuations on a set \( Y \). A dominant strategy mechanism \( m = (\tau, \pi) \) on \( \prod^n_i V_i \) is Pareto efficient on the set \( \mathcal{V} \subseteq \prod^n_i V_i \) if, for each \( \hat{v} \in \mathcal{V} \), there is a \( v \in \prod^n_i V_i \) such that

\[
\text{for each } i \in N, v_i \text{ is dominant for } i \text{ (with respect to } \hat{v}_i); \]

\[
m \text{ balances the budget at } v; \]

\[
\pi(v) \text{ maximizes } \varphi(\cdot, \hat{v}), \text{ the (true) aggregate valuation.} \]
The following remark is obvious.

**Remark 1:** A truth dominant mechanism is Pareto efficient on a set $V$ of profiles if and only if it is both budget balancing and valuation maximizing on $V$.

### 1.3. Groves Mechanisms

Groves [5] and Groves and Loeb [7] have discovered how to construct mechanisms which are truth dominant, when all admissible valuations attain a maximum. Namely, we first define the public decision function $\pi$ so as to always maximize the aggregate valuation; then for each $i \in N$ we arbitrarily define a function $h_i : \Pi_{j \neq i} V_j \to \mathbb{R}$; and finally we define each of the $n$ transfer functions as follows:

$$\tau_i(v) = \sum_{j \neq i} v_j(\pi(v)) - h_i(v_{-i}), \quad \forall v \in V.$$  

Mechanisms defined in this way are called Groves mechanisms. It is straightforward to verify that Groves mechanisms are indeed truth dominant, and combining this fact with Remark 1, we have the following remark.

**Remark 2:** A Groves mechanism on a product $\Pi_{1}^{n} V_i$ is Pareto efficient on a subset $V \subseteq \Pi_{1}^{n} V_i$ if and only if it balances the budget on $V$.

The following remark gives a useful alternative description of Groves mechanisms.

**Remark 3:** A mechanism $m = (\pi, \tau) : \Pi_{1}^{n} V_i \to Y \times \mathbb{R}^n$ admits $n$ functions $h_i : \Pi_{j \neq i} V_j \to \mathbb{R}$ $(i = 1, \ldots, n)$ which satisfy (10) if and only if the difference

$$\tau_i(v) - \sum_{j \neq i} v_j(\pi(v))$$

is always independent of $v_i$; i.e., if and only if $m$ satisfies the following condition:

**Condition G:** For each $i \in N$, each $v \in \Pi_{1}^{n} V_i$, and each $\tilde{v}_i \in V_i$:

$$\tau_i(v) - \sum_{j \neq i} v_j(\pi(v)) = \tau_i(v_{-i}, \tilde{v}_i) - \sum_{j \neq i} v_j(\pi(v_{-i}, \tilde{v}_i)),$$

which we will call the Groves condition. Thus, Groves mechanisms are precisely the ones which satisfy Condition G and are valuation maximizing on their entire domain.

Green and Laffont [3] and others [18, 21, 8] have shown that under certain fairly general assumptions about the admissible sets $V_1, \ldots, V_n$, the only mechanisms which satisfy Condition T are the ones which satisfy Condition G. It follows that the only mechanisms which are both Pareto efficient and truth dominant are the Groves mechanisms which always balance the budget. We will
show that (again, under fairly general assumptions about the sets $V_i$) no Groves mechanism can always balance the budget; hence, there is no mechanism which is both truth dominant and Pareto efficient. Since dominant strategy mechanisms can always be transformed into truth dominant ones without destroying Pareto efficiency (as we will describe in Section 4), it will follow that there can be no Pareto efficient dominant strategy mechanism.

The following remark, which is obtained by summing (10) over all $n$ players, gives a useful characterization of the budget balance condition for mechanisms which satisfy the Groves condition.

**Remark 5:** Suppose that the mechanism $m = (\pi, \tau): \Pi^n_i V_i \rightarrow Y \times \mathbb{R}^n$ satisfies Condition G. Let $h_i: \Pi_{i \neq i} V_i \rightarrow \mathbb{R}$ be as defined in (10) for each $i \in N$. Then $m$ balances the budget at the profile $v \in \Pi^n_i V_i$ if and only if

$$\sum_{i=1}^{n} h_i(v_{-i}) = (n - 1)\varphi(\pi(v), v).$$

2. **THE CUBICAL ARRAY LEMMA**

The essential tool in all of the proofs will be a criterion which must be satisfied whenever a mechanism satisfies both the Groves and budget balance conditions on a product $V_1 \times \ldots \times V_n$ in which all of the sets $V_i$ are doubletons or singletons. Because a product of doubletons contains $2^n$ elements, and because it is natural to think of each element as a corner or vertex of a cube, we will call such a product a **cubical array**. It will be important that one can generate cubical arrays from pairs of profiles, and we will want to have a convenient notation for doing so; we therefore make the following definition.

**Definition 6:** For any ordered pair $(v^1, v^2)$ of profiles on a set $Y$, the **cubical array** generated by $(v^1, v^2)$ is the set

$$\mathcal{C}(v^1, v^2) = \prod_{i=1}^{n} \{v_i^1, v_i^2\}.$$  

For each subset $S$ of $N$, the $S$ vertex of $\mathcal{C}(v^1, v^2)$ is the profile $v = (v_S, v_{-S})$ in which $v_i = v_i^1$ if $i \in S$ and $v_i = v_i^2$ if $i \not\in S$.

For an arbitrary product $\Pi^n_i V_i$ and for any given public decision function $\pi: \Pi^n_i V_i \rightarrow Y$, we can now define a criterion which can be used to characterize those cubical arrays in $\Pi^n_i V_i$ which admit budget balancing transfer functions satisfying Condition G.

**Definition 7:** Let $\mathcal{V} = \Pi^n_i V_i$ and let $\pi: \mathcal{V} \rightarrow Y$. The **budget balancing criterion** for $\pi$ is the function $F_{\pi}: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ defined by

$$F_{\pi}(v^1, v^2) = \sum_{S \subseteq N} (-1)^{|S|} \varphi(\pi(v^1_S, v^2_{-S}), (v^1_S, v^2_{-S})).$$
Let $\pi(S)$ denote the public decision at the $S$ vertex of the array $G(v^1, v^2)$,

$$\pi(S) = \pi(v_S, v_{-S}),$$

and let $\bar{\varphi}(S)$ denote the value of the aggregate valuation $\sum_{i \in S} v^1_i + \sum_{i \not\in S} v^2_i$ at $y = \pi(S)$,

$$\bar{\varphi}(S) = \varphi(\pi(S), (v_S, v_{-S})) = \sum_{i \in S} v^1_i (\pi(S)) + \sum_{i \not\in S} v^2_i(\pi(S)).$$

Then (13) can be written in the convenient form

$$F_\pi(v^1, v^2) = \sum_{S \subseteq N} (-1)^{\#S} \bar{\varphi}(S).$$

Now we come to the Cubical Array Lemma, which states that a public decision function $\pi$ on a cubical array will admit transfer functions which satisfy the Groves and budget balance conditions only if $F_\pi$ has the value zero on the array.

**The Cubical Array Lemma:** Let $Y$ be a set; let $n \geq 2$; and let $v^1$ and $v^2$ be two $n$-profiles on $Y$. If a mechanism $m = (\pi, \tau)$ satisfies Condition $G$ and balances the budget on the cubical array $G(v^1, v^2)$, then $F_\pi(v^1, v^2) = 0$.

**Proof:** We have

$$F_\pi(v^1, v^2) = \sum_{S \subseteq N} (-1)^{\#S} \varphi(\pi(v^1_S, v^2_{-S}), (v^1_S, v^2_{-S}))$$

$$= \sum_{S \subseteq N} (-1)^{\#S} \left( \frac{1}{n-1} \right) \sum_{i \in N} h_i(v^1_{S \setminus \{i\}}, v^2_{-S \setminus \{i\}}), \quad \text{by Remark 5,}
$$

$$= \frac{1}{n-1} \sum_{i \in N} \sum_{S \subseteq N} (-1)^{\#S} h_i(v^1_{S \setminus \{i\}}, v^2_{-S \setminus \{i\}})$$

$$= \frac{1}{n-1} \sum_{i \in N} \left[ \sum_{S \in \mathcal{E}} h_i(v^1_{S \setminus \{i\}}, v^2_{-S \setminus \{i\}}) - \sum_{S \in \mathcal{O}} h_i(v^1_{S \setminus \{i\}}, v^2_{S \setminus \{i\}}) \right]$$

where $\mathcal{E} = \{S \subseteq N | \# S \text{ is even}\}$ and $\mathcal{O} = \{S \subseteq N | \# S \text{ is odd}\}$. We will show that the expression in brackets is zero for each $i \in N$.

Let $i \in N$, and let $i$ be fixed for the remainder of the proof. We partition the sets $\mathcal{E}$ and $\mathcal{O}$ into those sets which contain $i$ and those which do not not:

$$\mathcal{E}_0 = \{S \subseteq \mathcal{E} | i \notin S\}, \quad \mathcal{E}_1 = \{S \subseteq \mathcal{E} | i \in S\}$$

$$\mathcal{O}_0 = \{S \subseteq \mathcal{O} | i \notin S\}, \quad \mathcal{O}_1 = \{S \subseteq \mathcal{O} | i \in S\}.$$

Two lemmas will be needed to complete the proof. The first is obvious.

**Lemma A:** The map from $\mathcal{E}_1$ to $\mathcal{O}_0$ which carries $S$ to $S \setminus \{i\}$ is a one-to-one correspondence between $\mathcal{E}_1$ and $\mathcal{O}_0$. The map from $\mathcal{E}_0$ to $\mathcal{O}_1$ which carries $S$ to $S \cup \{i\}$ is a one-to-one correspondence between $\mathcal{E}_0$ and $\mathcal{O}_1$. 
For each \( S \subseteq N \), we abbreviate the \((n - 1)\)-tuple \((v_{S|\{i\}}, v_{S|\{i\}}^2)\), denoting it simply by \(w_S\). We have the following lemma.

**Lemma B:** For each \( S \subseteq N \), \(w_{S|\{i\}} = w_S = w_{S \cup \{i\}}\).

**Proof:** For the \(v^1\) components of the three \((n - 1)\)-tuples:
\[
(S \setminus \{i\}) \setminus \{i\} = S \setminus \{i\} = (S \cup \{i\}) \setminus \{i\}.
\]
For the \(V^2\) components:
\[
\neg(S \setminus \{i\}) \setminus \{i\} = (\neg S \cup \{i\}) \setminus \{i\}
\]
\[
= \neg S \setminus \{i\}
\]
\[
= (\neg S \setminus \{i\}) \setminus \{i\}
\]
\[
= \neg (S \cup \{i\}) \setminus \{i\}.
\]
and Lemma B is proved.

Now the proof that \(\sum_{S \in \Sigma} h_i(w_S) - \sum_{S \in \emptyset} h_i(w_S) = 0\) is easy. We have
\[
\sum_{S \in \Sigma} h_i(w_S) = \sum_{S \in \Sigma_1} h_i(w_S) + \sum_{S \in \Sigma_0} h_i(w_S)
\]
\[
= \sum_{S \in \Sigma_0} h_i(w_{S|\{i\}}) + \sum_{S \in \Sigma_1} h_i(w_{S|\{i\}}), \text{ by Lemma A,}
\]
\[
= \sum_{S \in \Sigma_0} h_i(w_S) + \sum_{S \in \Sigma_1} h_i(w_S), \text{ by Lemma B,}
\]
\[
= \sum_{S \in \emptyset} h_i(w_S).
\]
Q.E.D.

Whenever a function \(\tau\) is defined on a cubical array \(\mathcal{C}(v^1, v^2)\), we will say that both the array and the pair \((v^1, v^2)\) are balanced if \(F_\tau(v^1, v^2) = 0\), and that they are unbalanced if \(F_\tau(v^1, v^2) \neq 0\). We have just shown that arrays which are unbalanced do not admit Groves transfer functions which balance the budget. The converse—that every balanced array admits a budget-balancing mechanism satisfying Condition G (on the array)—can also be shown, but we will not need that result.

3. Theorems for Groves-type Mechanisms

We are now ready for the central theorem, which states that Groves mechanisms cannot generally balance the budget. For certain parametric classes of environments (e.g., see [19], where quadratic valuations are treated) one can exhibit specific unbalanced cubical arrays (for the valuation maximizing public decision function \(\tau\)); this guarantees, according to the Cubical Array Lemma, that there will always be profiles on which Groves mechanisms fail to balance the budget. Suppose, however, that we could find a Groves mechanism for which such optimality failure was very rare—namely, a mechanism which failed to attain
optimal outcomes on only a negligible proportion of all possible profiles. Then the negative result would not be of much concern. Let us concentrate our attention then on a more sweeping result: that any Groves mechanism will virtually always attain nonoptimal outcomes.

More precisely, Theorem 1 will state that if \( m \) satisfies Condition G, then for any profile \( v \) which is not (topologically) isolated, there are profiles arbitrarily close to \( v \) (in a well defined sense) at which \( m \) does not balance the budget. The following notation will be helpful.

For any mechanism \( m : \Pi^n V_i \rightarrow \mathbb{R} \times Y \), let us define the **success set** of \( m \), denoted \( \mathcal{S}(m) \), to be the set of profiles in \( \Pi^n V_i \) at which \( m \) yields a Pareto optimal outcome, and the **failure set**, denoted \( \mathcal{F}(m) \), to be the complement of \( \mathcal{S}(m) \) in \( \Pi^n V_i \)—the set of profiles at which \( m \) yields nonoptimal outcomes. Theorem 1 states that if \( Y \) is a convex open set, if each \( V_i \) is the set of all strictly concave maximum attaining valuations on \( Y \), and if \( m \) satisfies Condition G, then the failure set \( \mathcal{F}(m) \) is everywhere dense in \( \Pi^n V_i \).

Beginning with Theorem 1, nearly all of the remaining results are topological, so we must be explicit about which topology(s) on \( \Pi^n V_i \) we are going to consider. Thus, for any set \( Y \) in \( \mathbb{R}^\ell \), let \( C(Y) \) denote the set of all continuous real-valued functions on \( Y \); each set \( V_i \) of valuations will be a subset of \( C(Y) \), and \( \Pi^n V_i \) will thus be a subset of \( [C(Y)]^n \). The **vector-space topology** on any \( V_h \), denoted by \( \mathcal{F} \), is the one it inherits as a subspace of \( C(Y) \), when the latter is endowed with the largest topology in which it is a topological vector space—i.e., the largest topology with respect to which both addition and scalar multiplication (each defined pointwise) are continuous operations.

Theorem 1—that the failure set \( \mathcal{F}(m) \) is dense in \( \Pi^n V_i \)—is proved for the vector-space topology and is thus true for all smaller topologies as well (i.e., for topologies \( \mathcal{T} \subseteq \mathcal{F} \)). In subsequent results, when we ask for a mechanism that is continuous, the use of such a large topology as \( \mathcal{F} \) guarantees that we are casting our net relatively far: the larger the topology the less restrictive is the continuity assumption, and continuity with respect to \( \mathcal{F} \) is about as weak a continuity restriction as one would ever want.

**Theorem 1:** Let \( n \geq 2 \); let \( Y \) be a convex open set in \( \mathbb{R}^\ell \); let \( V \) be the set of all strictly concave maximum attaining valuations on \( Y \); and let \( m = (\tau, \pi) \) be a mechanism on \( V^n \). If \( m \) satisfies Condition G and if the operations of addition and scalar multiplication in \( V^n \) are continuous (i.e., if \( V \) carries a topology \( \mathcal{T} \subseteq \mathcal{F} \)), then the failure set \( \mathcal{F}(m) \) is everywhere dense in \( V^n \).

**Proof:** Let \( v \in V^n \); we must show that each neighborhood of \( v \) contains a profile at which either (5) or (6) fails. We assume that both (5) and (6) hold at \( v \), and that there is a neighborhood of \( v \) on which (5) always holds (otherwise there is nothing left to prove.) We must show that each neighborhood of \( v \) contains a profile at which (6) fails—i.e., at which the budget is not balanced.

Actually, we will show something a bit stronger: that every basic neighborhood contains a profile at which the budget is unbalanced. Since \( \mathcal{F}^{(n)} \) is a product topology, its basic neighborhoods are of the form \( \mathcal{N}^{(n)} = \mathcal{N}_1 \times \ldots \times \mathcal{N}_n \), where (for
each $i \in \mathcal{N}$) $\mathcal{N}_i$ is a neighborhood in $V$, with respect to $\bar{F}$. Consequently, if two profiles $v$ and $w$ lie in a basic neighborhood $\mathcal{N}$, then the cubical array $\mathcal{C}(v, w)$ also lies in (i.e., is a subset of) $\mathcal{N}$. Our method of proof, then, will be to show that the criterion $F_\pi$ defined in (13) is never locally constant; in particular, every basic neighborhood $\mathcal{N}$ of a profile $v$ contains a profile $w$ such that $F_\pi(v, w) \neq 0$. Since $\mathcal{C}(v, w) \subseteq \mathcal{N}$, the Cubical Array Lemma will then guarantee that $\mathcal{N}$ contains a profile at which the budget is unbalanced.

Let $\mathcal{N} = \mathcal{N}_1 \times \ldots \times \mathcal{N}_n$ be a basic neighborhood of $v$, and let $w \in \mathcal{N}$. Recall from (16) that

$$F_\pi(v, w) = \sum_{S \in \mathcal{N}} (-1)^{\#S} \varphi(S).$$

We will perturb the pair $(v, w)$ to a new pair in $\mathcal{N} \times \mathcal{N}$, in such a way that precisely one of the numbers $\varphi(S)$ is changed, thereby changing the value of $F_\pi$. This perturbation is carried out in Appendix A.

Q.E.D.

A plausible conjecture in light of Theorem 1 is that any mechanism satisfying Condition G will have a success set which is nowhere dense in $V^n$. I have no idea whether that conjecture is true or false. However, if we consider continuous mechanisms—those in which the outcomes $(x, y)$ vary continuously with respect to the profiles—then the conjecture is true: the failure set is open, as well as dense, so its complement (the success set) is closed and nowhere dense. This is the content of Theorem 2.

Since we have already demonstrated in Theorem 1 that $\mathcal{F}(m)$ is dense in $V^n$ for any topology as coarse (i.e., as small) as $\bar{F}$ on the set $V$, we can obtain Theorem 2 merely by proving a lemma to the effect that $\mathcal{F}(m)$ is open, or equivalently, that $\mathcal{I}(m)$ is closed.\footnote{The proof of Theorem 2 and all subsequent results stating that "$\mathcal{I}(m)$ is closed" are valid for any topology on the sets $V_i$ which is as coarse (i.e. small) as the vector-space topology and as fine (i.e. large) as the topology of compact convergence (also called the topology of uniform convergence on compact sets, and equivalent here to the compact-open topology; see, e.g., Kelley [11, Chapter 7]).}

**Theorem 2:** Let $n \geq 2$; let $Y$ be a convex open set in $\mathbb{R}^d$; let $V$ be the set of all strictly concave maximum attaining valuations on $Y$; and let $V^n$ be endowed with the vector-space topology $\bar{F}$. If $m$ is a continuous mechanism on $V^n$ which satisfies Condition G, then the success set $\mathcal{I}(m)$ is closed and is nowhere dense in $V^n$.

As described above, the theorem is an immediate consequence of the following lemma.

**Lemma:** Under the conditions of Theorem 2, the success set $\mathcal{I}(m)$ is closed.

**Proof:**

$$\mathcal{I}(m) = \left\{ v \mid \sum_{i=1}^{n} \tau_i(v) = 0 \right\} \cap \{ v \mid \pi(v) \text{ is the maximizer of } \varphi(\cdot, v) \}.$$
The first set in this intersection is closed, because each $\tau_i$ is continuous. The second set is closed because $\pi$ is continuous, and because the function which assigns to each $v$ the maximizer of $\varphi(\cdot, v)$ is continuous (this is fairly straightforward; see, for example, [20]). 

Q.E.D.

4. THEOREMS FOR DOMINANT STRATEGY MECHANISMS

It was shown in Walker [18, 21] that any truth dominant mechanism must satisfy Condition G if the admissible sets $V_i$ together satisfy the following “richness” condition:

\[(*) \quad \text{For each } i \in N; \text{ each } v_{-i} \in \prod_{j \neq i} V_j; \text{ each } y, \tilde{y} \in \pi(V_i \times \{v_{-i}\}); \text{ and each } \varepsilon > 0: \text{ there is a } v_i \in V_i \text{ such that}
\]

\[
(i) \quad \pi(v_i, v_{-i}) = y,
\]

and

\[
(ii) \quad v_i(y) + \sum_{j \neq i} v_j(y) \leq v_i(\tilde{y}) + \sum_{j \neq i} v_j(\tilde{y}) + \varepsilon.
\]

If each $V_i$ is the set of all strictly concave maximum attaining valuations on a convex set $Y$, then $(*)$ is clearly satisfied. Thus, Theorems 1 and 2 immediately yield the following theorem.

**Theorem 3:** Theorems 1 and 2 remain true if “Condition G” is replaced by “Condition T” (i.e., the two theorems are true for truth dominant mechanisms.)

Now let us consider mechanisms which have the dominant-strategy property, but which are not truth dominant. Let $Y$ be a convex set; let $V_i$ be a set of valuations on $Y$, for each $i \in N$; and let $m$ be a dominant-strategy mechanism on $\Pi_i V_i$. For each $i \in N$ and each $v_i \in V_i$, let $\delta_i(v_i)$ denote a dominant strategy for $i$ (with respect to $v_i$ and $m$). We have $n$ functions $\delta_i : V_i \to V_i$ ($i = 1, \ldots, n$). Let $\delta = \delta_1 \times \ldots \times \delta_n : \Pi_i V_i \to \Pi_i V_i$. Now the mechanism $\mathfrak{m}$ defined by $\mathfrak{m}(v) = m(\delta(v))$ is clearly truth dominant. In other words, if there is a dominant strategy mechanism on $\Pi_i V_i$, then there is also a truth dominant mechanism on $\Pi_i V_i$. The following theorem is therefore a straightforward consequence of Theorem 1.

**Theorem 4:** Let $n \geq 2$; let $Y$ be a convex open set in $\mathbb{R}^r$, and (for each $i \in N$) let $V_i$ be the set $V$ of all strictly concave maximum attaining valuations on $Y$. Then there is no Pareto efficient dominant strategy mechanism on $\Pi_i V_i$; that is, any dominant strategy mechanism on $\Pi_i V_i$ must have a nonempty failure set $\mathcal{F}(m)$.

Stronger descriptions of the sets $\mathcal{H}(m)$ and $\mathcal{F}(m)$—analogues of Theorems 1, 2, and 3—are more difficult, because the functions $\delta_i$ may, in general, be discontinuous and/or multi-valued. A straightforward application of the Maximum
Theorem will verify that each $\delta_i$ (and thus $\delta$) has a closed graph if $m$ is a continuous mechanism, but unless the sets $V_i$ are compact, the closed graph property seems to be of little use. When the $V_i$ are compact, we have the following.

**Theorem 5:** Let $n \geq 2$; let $Y$ be a convex open set in $\mathbb{R}^e$; and let the set of continuous functions $C(Y)$ have the vector-space topology. If each set $V_i$ is a compact subset of $C(Y)$ and $m$ is a continuous dominant strategy mechanism on $\Pi_1^n V_i$, then the success set $\mathcal{F}(m)$ is closed and nowhere dense in $\Pi_1^n V_i$.

5. Applications

All of the theorems that have been presented here deal directly with situations in which there are only one private good and several pure public goods, and in which alternative levels of public good provision do not involve differential resource costs. Further, any lower bounds on individuals' private good assignments have been ignored. It is straightforward, however, to apply Theorems 3, 4, and 5 to obtain analogous results when there are differential costs to alternative public good levels, and when explicit account is taken of lower bounds on consumption, and (if there are only two players) when both goods are private ones. Such results show, in particular, that the impossibility results of Hurwicz [9] for private goods and Ledyard and Roberts [12] for one public and one private good are both generic in character. Similar results can almost certainly be proved when there are more goods (and more players in the private goods case), and when there are other sorts of externalities. Instead of directly applying the theorems here, however, one must obtain Theorem 1 for the many-goods case; the only alteration required in the proof given here for Theorem 1 is that the result in Appendix B—that the existence of an isolated set is a dense property—is more difficult to prove.

6. Concluding Remarks

The results that have been presented here indicate that we cannot generally have dominant strategies and Pareto optimality together. However, this certainly does not rule out the existence of perfectly good mechanisms. If one is willing to expend the necessary resources, for example, then we know that the mechanisms discovered by Groves do enable one to elicit truthful behavior as a dominant strategy.

The first question to suggest itself then is whether there are dominant strategy mechanisms which, although not achieving Pareto optimal outcomes, nevertheless keep the resource loss small. In other words, are there dominant strategy mechanisms which always yield outcomes that are very nearly Pareto optimal? So far as I know, the only analysis of this sort of question is the one by Green, Kohlberg, and Laffont [2], whose results suggest that we may be able to keep the

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4 This is treated in Working Paper No. 211 of the Economic Research Bureau at S.U.N.Y. Stony Brook, which is a somewhat longer version of this paper, with the same title.
expected resource cost small by subjecting only a small, randomly selected sample of the population to a particular Groves mechanism. This is a promising idea and deserves further attention.

The opposite approach—retention of exact optimality and abandonment of the dominant strategy requirement—was first taken, with considerable success, by Groves and Ledyard [6]. They devised an explicit mechanism for allocating both private and public goods which does not yield dominant strategies, but in which the Nash equilibria are always Pareto optimal. Their approach has been followed by numerous others, who have, for the most part, devised alternative mechanisms and investigated the mechanisms' Nash equilibria.

A third approach derives from the work of Roberts and Postlewaite [16], who demonstrated that, in large classical private goods economies, Hurwicz’s impossibility result (in [9]; described in the introduction to this article) can be expected to vanish, in a certain sense. When the price mechanism is used in a large economy, the individual’s gain from employing misleading behavior is at best miniscule (and, if he should make the slightest error, the gain will become a loss). Thus, price-taking (or sincere) behavior can be viewed as a sort of “asymptotically dominant” strategy; and of course the price mechanism’s equilibria are well known to be Pareto optimal in nice private goods economies.

When public goods are present the Roberts-Postlewaite result no longer holds. Indeed, Roberts [15] has shown that no mechanism in a fairly general class of mechanisms can exhibit the “asymptotic dominance property” that the price mechanism displays when there are only private goods. It seems, moreover, that the mechanism of Groves and Ledyard may be subject to certain kinds of failure in large economies (see Muench and Walker [14]). Whether there is any mechanism which can attain “good” outcomes in large public goods economies is still an open question.

Finally, this emphasis on pessimistic results when there are public goods will apparently have to be taken into account when we evaluate alternative mechanisms. Since it appears that there may not be any mechanism which is successful in every respect, we will need a theory in which we can deal analytically with the trade-offs among various kinds of failure. A formal treatment involving certain trade-offs of this kind can be found in the work of Muench and Walker [13, 14].

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APPENDIX A

To complete the proof of Theorem 1 we must perturb the pair \((v, w)\) in such a way that precisely one of the numbers \(\varphi(S)\) in (39) is changed. We will have to decide which subset of \(N\)—denote it by \(S'\)—will be the one for which \(\varphi(S')\) is altered. Let this set be any \(S'\) which is isolated, in the sense that

\[
\pi(S) \neq \pi(S') \quad \text{for all} \quad S \neq S',
\]
where \( \pi(S) \) is the public decision as defined in (14), which is, in turn, the maximizer of the function \( \Sigma_{v_i} + \Sigma_{w_i} \) because (5) is satisfied on \( \mathcal{N} \). (It is shown in Appendix B that if there is no isolated set, we can perturb \( v \) or \( w \) by an arbitrarily small amount (with respect to \( \mathcal{T} \)) so as to obtain an isolated set.) For our isolated set \( S' \), we denote \( \pi(S') \) by \( \tilde{y} \)—i.e., \( \tilde{y} \) is the unique maximizer of the function

\[
\left( \sum_{i \in S'} v_i + \sum_{i \not\in S'} w_i \right) : Y \rightarrow \mathbb{R}.
\]

Because \( S' \) is isolated, there is a closed ball \( B \) of positive radius about \( \tilde{y} \) such that \( B \) contains none of the other maximizers \( \pi(S) \) for \( S \neq S' \).

Since \( \tilde{y} \) is a minimizer of the function \( \Sigma_{S'} v_i + \Sigma_{-S'} w_i \), there are vectors \( s_1, s_2, \ldots, s_n \in \mathbb{R}^m \) such that

\[
\sum_{i=1}^{n} s_i = 0
\]

and

\[
\begin{aligned}
& \{ s_i \text{ is a supergradient of } v_i \text{ at } \tilde{y} \text{ if } i \in S', \\
& \{ s_i \text{ is a supergradient of } w_i \text{ at } \tilde{y} \text{ if } i \not\in S'.
\end{aligned}
\]

Let \( i \) be any player, say in the set \( S' \). We will construct a new valuation \( \tilde{v}_i \) which has the following properties:

\[
\begin{aligned}
& \text{(44)} \quad s_i \text{ is a supergradient of } \tilde{v}_i \text{ at } \tilde{y} \text{ (where } s_i \text{ is defined by (42) and (43))}; \\
& \text{(45)} \quad \tilde{v}_i(\tilde{y}) < v_i(\tilde{y}); \\
& \text{(46)} \quad \tilde{v}_i(y) \leq v_i(y) \text{ for all } y \in B; \\
& \text{(47)} \quad \tilde{v}_i(y) = v_i(y) \text{ for all } y \not\in B; \\
& \text{(48)} \quad \tilde{v}_i \text{ strictly concave}; \\
& \text{(49)} \quad \tilde{v}_i \in \mathcal{N}.
\end{aligned}
\]

(We could in fact perturb any or all of the valuations in the same way, with \( w_i \) replacing \( v_i \) if \( i \not\in S' \).) Conditions (46), (47), and (48) ensure that \( \varphi(S) \) is unchanged by the perturbation of \( v_i \) to \( \tilde{v}_i \) if \( S \neq S' \).

Let \( \tilde{y} \) be the new maximizer for \( S' \), i.e. \( \tilde{y} \) maximizes the function

\[
\tilde{v}_i + \sum_{i \in S'} v_i + \sum_{i \not\in S'} w_i;
\]

and let \( \tilde{\varphi}(S') \) be the value of the function (50) at \( \tilde{y} \)—i.e., \( \tilde{\varphi}(S') \) is our new value of \( \varphi(S') \). Conditions (44) and (48) ensure that \( \tilde{y} = \tilde{y} \), and (45) insures that \( \tilde{\varphi}(S') \leq \varphi(S') \), as desired.

The following construction yields a function \( \tilde{\varphi} \) which satisfies (44)–(49). First, let \( B' \) be the boundary of \( B \). Notice that \( B' \) is compact and that \( \tilde{y} \in B' \). Next, for each \( z \in B' \) and each \( y \in Y \), let

\[
f(y, z) = v_i(z) + s_i \cdot (y - z).
\]

For each \( z \in B' \), the function \( f(y, z) \) on \( Y \) is the affine function which has gradient \( s_n \) and the graph of which passes through \( (z, v_i(z)) \in \mathbb{R}^{m+1} \). Notice that

\[
f(z, z) = v_i(z) \quad \text{for each } z \in B';
\]

Notice too that

\[
\begin{aligned}
& v_i(\tilde{y}) > f(\tilde{y}, z) \quad \text{for each } z \in B', \\
& v_i(\tilde{y}) \text{ is strictly concave and } \tilde{y} \not\in B'.
\end{aligned}
\]

Now consider the function \( f(\tilde{y}, \cdot) \) on the compact set \( B' \). Since this function is continuous, we may choose \( \tilde{\varepsilon} \in B \) to be a maximizer of the function. Let \( g: Y \rightarrow \mathbb{R} \) be the function defined by

\[
g(y) = \min \{ v_i(y), f(y, \tilde{\varepsilon}) \}.
\]

Figure 1 depicts the construction so far, in the case \( \ell = 1 \) (in which case \( B \) is a closed interval and \( B' \) is a doubleton.) Notice that \( g \) is a concave function.

Finally, let us define \( \tilde{v}_i = Y \rightarrow \mathbb{R} \) as follows:

\[
\tilde{v}_i = (1 - \lambda) v_i + \lambda g,
\]
where $0 < \lambda < 1$. The function $\tilde{v}_i$ clearly satisfies (44)–(46) and (48). By choosing $\lambda$ sufficiently near zero, we can clearly insure that $\tilde{v}_i \in \mathcal{N}_i^*$ (because the operations of addition and scalar multiplication are continuous in $C(Y)$ with respect to $\mathcal{F}$). This leaves us to verify only (47), that $\tilde{v}_i$ and $v_i$ agree on $Y \setminus B$.

Suppose that $\tilde{v}_i(y) \neq v_i(y)$ for some $y \notin B$—i.e. according to the definition of $\tilde{v}_i$ in (55),

$$\tilde{v}_i(y) > f(y, \hat{\hat{z}}).$$

We also have, as a special case of (53),

$$v_i(y) > f(y, \hat{\hat{z}}).$$

Because $v_i$ is concave and $f(\cdot, \hat{\hat{z}})$ is affine, it follows from (56) and (57) that

$$v_i(z) > f(z, \hat{\hat{z}})$$

for each $z$ on the line segment joining $y$ and $\hat{\hat{y}}$. Since $\hat{\hat{y}} \in B$ and $y \notin B$, one such $z$ lies in $B^*$; for that $z$, we combine (52) and (58) to obtain

$$f(z, z) > f(z, \hat{\hat{z}}).$$

Applying the definition (51) of $f$, (59) yields

$$v_i(z) > v_i(\hat{\hat{z}}) + s_i \cdot (z - \hat{\hat{z}}).$$

But the definition of $\hat{\hat{z}}$ yields

$$f(\hat{\hat{y}}, \hat{\hat{z}}) = f(\hat{\hat{y}}, z),$$

and if we similarly apply the definition of $f$ to (61), we obtain

$$v_i(\hat{\hat{z}}) + s_i (z - \hat{\hat{z}}) \geq v_i(z).$$

Combining (60) and (62), we have $v_i(z) > v_i(z)$, a contradiction which completes the proof. \[ Q.E.D. \]
It will be shown here that if the pair \((v, w)\) in the proof of Theorem 1 does not yield an isolated set (as defined in Appendix A), there will nevertheless be a pair arbitrarily near \((v, w)\) which does yield an isolated set. We will take each \(v_i\) and \(w_i\) to be twice continuously differentiable, which is sufficient since the set of all such functions is dense in the set \(V\) (with respect to the topology \(\mathcal{F}\)).

Let the \(n\)-tuple \(v\) be fixed throughout; we will choose each \(w_i\) to be a linear perturbation of \(v_i\):

\[
w_i(y) = v_i(y) + \lambda \sum_{k=1}^{i} \alpha_{ik} y_k
\]

where \(\lambda\) will be chosen near zero, and we will be free to choose the coefficients \(\alpha_{ik}\). The following notation will be helpful:

\[
\text{(63) } \phi(y, S, \lambda) = \sum_{i \in S} v_i(y) + \sum_{i \notin S} w_i(y) \text{ for each } (y, S, \lambda);
\]

\[
\text{(64) } \phi(y(S, \lambda)) \text{ denote the maximizer of } \phi(\cdot, S, \lambda) \text{—i.e., that } y \text{ at which } \frac{\partial \phi}{\partial y_k} = 0 \text{ for each } k, \text{ since } Y \text{ is open.}
\]

Our objective will be to show that the coefficients \(\alpha_{ik}\) can be chosen in such a way that, for some nonempty subset \(S'\) of \(N\),

\[
\text{(65) } \frac{\partial \phi(y, S, \lambda)}{\partial \lambda} \neq \frac{\partial \phi(y(S', \lambda))}{\partial \lambda} \text{ for all } S \neq S' \text{ and all } k = 1, \ldots, \ell.
\]

Then, by choosing \(\lambda\) small enough, we will have both \(w \in \mathcal{N}\) (where \(\mathcal{N}\) is the given neighborhood of \(v\)), and \(y(S, \lambda) \neq y(S', \lambda)\) for all \(S \neq S'\)—i.e., \(S'\) will be isolated.

To obtain an expression for \(\frac{\partial \phi(y, S, \lambda)}{\partial \lambda}\) we apply the Implicit Function Theorem to the system of equations

\[
\text{(66) } \frac{\partial \phi(y, S, \lambda)}{\partial y_k} = 0 \quad (k = 1, \ldots, \ell).
\]

Since

\[
\frac{\partial \phi(y, S, \lambda)}{\partial y_k} = \sum_{i \in S} \frac{\partial v_i}{\partial y_k} + \sum_{i \notin S} \frac{\partial w_i}{\partial y_k} = \sum_{i \in S} \frac{\partial v_i}{\partial y_k} + \lambda \sum_{i \in S} \alpha_{ik} = \frac{\partial \phi(y, S, 0)}{\partial y_k} + \lambda \sum_{i \in S} \alpha_{ik},
\]

we obtain the following system of differential equations:

\[
\sum_{i \in S} \alpha_{ik} d\lambda + \sum_{j=1}^{\ell} \frac{\partial \phi(y, S, 0)}{\partial y_j} dy_j = 0 \quad (k = 1, \ldots, \ell).
\]

Let \(\Phi\) denote the \(\ell \times \ell\) matrix of terms \(\frac{\partial \phi(y, S, 0)}{\partial y_j} \partial y_k\); note that \(\Phi\) is independent of \(S\), and is negative definite (because each \(v_i\) is strictly concave). The Implicit Function Theorem yields, for each \(S\),

\[
\text{(67) } \frac{dy_k(S, 0)}{\partial \lambda} = -\Phi^{-1} \sum_{i \in S} \alpha_{ik} \quad (k = 1, \ldots, \ell).
\]

Now we can use (67) to make judicious choice of the coefficients \(\alpha_{ik}\)—a choice that will satisfy (65) for some set \(S'\). First choose an arbitrary player \(\ell \in N\), and let \(S' = N \setminus \{\ell\}\). Let \(e\) be an arbitrary strictly positive vector in \(\mathbb{R}^\ell\), and choose the coefficients \(\alpha_{ik}\) as follows: for each \(i \neq \ell\), let \(\alpha_i = \Phi e\); and let
\[ \alpha_t = \frac{1}{2} \Phi e. \] Thus, for each set \( S \) we have

\[ \Phi^{-1} \sum_{i \in S} \alpha_{ik} = \begin{cases} (n - \# S)e, & \text{if } t \in S, \\ (n - \# S - \frac{1}{2})e, & \text{if } t \notin S. \end{cases} \]

In particular,

\[ \Phi^{-1} \sum_N \alpha_{ik} = 0, \]

\[ \Phi^{-1} \sum_S \alpha_{ik} = \frac{1}{2} e, \]

and

\[ \Phi^{-1} \sum_S \alpha_{ik} \geq e \quad \text{for all other } S. \]

Finally, the components of \( e \) can now clearly be chosen in such a way that the system (67) satisfies (65).

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