Economics 501B Midterm Exam Solutions
Fall 2017

1. We will give an indirect proof by proving the contrapositive statement: If $(\widehat{\mathbf{x}}^i)_N$ is not a Pareto allocation then there is some $h \in N$ for which $(\widehat{\mathbf{x}}^i)_N$ is not a solution of P-Max[h]. Assume that $(\widehat{\mathbf{x}}^i)_N$ is feasible — *i.e.*, $\sum_{i \in N} \widehat{\mathbf{x}}^i \leqq \mathring{\mathbf{x}}$ — but is not a Pareto allocation. Then there is a Pareto improvement on $(\widehat{\mathbf{x}}^i)_N$ — a feasible allocation $(\widetilde{\mathbf{x}}^i)_N$ that satisfies

$$(1) \quad \forall i \in N : u^i(\widetilde{\mathbf{x}}^i) \geqq u^i(\widehat{\mathbf{x}}^i) \qquad \text{and} \qquad (2) \quad \exists h \in N : u^h(\widetilde{\mathbf{x}}^h) > u^h(\widehat{\mathbf{x}}^h).$$

Because $(\widetilde{\mathbf{x}}^i)_N$ is feasible and satisfies (1), it satisfies all the constraints of every one of the problems P-Max[h]. Therefore (2) implies that there is an $h \in N$ such that $(\widehat{\mathbf{x}}^i)_N$ is not a solution of P-Max[h]. ■

② $u^i(x,y) = xy \quad (i=A,B) \qquad MRS^i = \frac{y_i}{x_i}$

$(\mathring{x}_A, \mathring{y}_A) = (1,9), \quad (\mathring{x}_B, \mathring{y}_B) = (9,1); \quad \mathring{u}_A = \mathring{u}_B = 9.$

(a) <u>PARETO:</u> $MRS^A = MRS^B$, i.e., $\frac{y_A}{x_A} = \frac{y_B}{x_B} = \frac{10-y_A}{10-x_A}$ $\quad \therefore \sqrt{u_A}+\sqrt{u_B} = 10$

$\therefore 10y_A - \cancel{x_A y_A} = 10x_A - \cancel{x_A y_A}$; i.e., $10y_A = 10x_A$, i.e., $x_A = y_A$

$\therefore x_A = y_A,\ x_B = y_B,\ x_A + x_B = 10,\ y_A + y_B = 10; \quad u_A = x_A^2,\ u_B = x_B^2.$

(b) <u>WALRASIAN EQUIL'M:</u> $p_x = p_y$ AND $(x_A, y_A) = (x_B, y_B) = (5,5)$;
$u_A = u_B = 25.$

VERIFYING: (ASSUME $p_x = p_y = 1$)

(M-CLR) $(x_A, y_A) + (x_B, y_B) = (10,10) = (\mathring{x}, \mathring{y})$.

(U-MAX) $p_x x_A + p_y y_A = x_A + y_A = 10 = p_x \mathring{x}_A + p_y \mathring{y}_A = 1+9$

~~SAME FOR B~~ AND $MRS^A = \frac{5}{5} = 1 = \frac{p_x}{p_y}$.

SAME FOR B.

(c) <u>CORE:</u>

PARETO IMPLIES $x_A = y_A,\ x_B = y_B,\ \therefore u_A = x_A^2,\ u_B = x_B^2$.

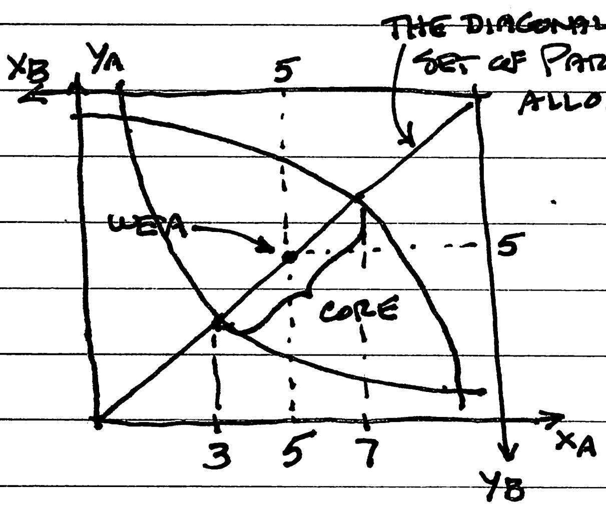
INDIVIDUAL RATIONALITY REQUIRES $u_A \geqq \mathring{u}_A = 9,\ u_B \geqq \mathring{u}_B = 9$.

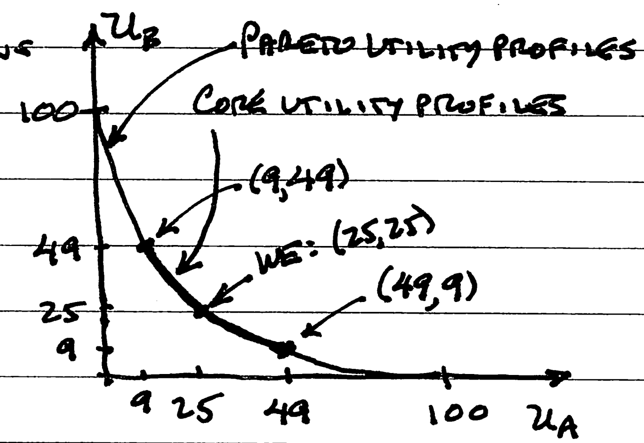
$\therefore x_A^2, x_B^2 \geqq 9$, i.e., $x_A, x_B \geqq 3$.

SO THE CORE ALLOCATIONS ARE THE ONES THAT SATISFY

$x_A = y_A,\ x_B = y_B,\ 3 \leqq x_A \leqq 7,\ 3 \leqq x_B \leqq 7,\ x_A + x_B = y_A + y_B = 10.$

IN UTILITY TERMS: $9 \leqq u_A \leqq 49,\ 9 \leqq u_B \leqq 49,\ \sqrt{u_A} + \sqrt{u_B} = 10$.





ADD $u^c(x,y)=xy$, $(\mathring{x}_c, \mathring{y}_c)=(5,5)$, $\mathring{u}_c=25$.

(d) <u>PARETO:</u> $MRS^A = MRS^B = MRS^C$, $\therefore x_i = y_i \ (i=A,B,C)$

$$\text{AND } \sum x_i = \mathring{x} = 15, \quad \sum y_i = \mathring{y} = 15.$$

$$\therefore \sqrt{u_A} + \sqrt{u_B} + \sqrt{u_C} = 15.$$

(e) <u>WALRASIAN EQUIL'M:</u> $p_x = p_y$; $(x_i, y_i) = (5,5)$, $i=A,B,C$.

$$\therefore u_A = u_B = u_C = 25.$$

(f) <u>CORE:</u>

$S=N$ (PARETO): $x_i = y_i \ (\forall i)$ FROM (d), $\therefore \sqrt{u_A}+\sqrt{u_B}+\sqrt{u_C}=15$.

$S=\{i\}$: $u_A \geqq \mathring{u}_A = 9$, $\therefore \sqrt{u_A} \geqq 3$, $\therefore x_A, y_A \geqq 3$

$u_B \geqq \mathring{u}_B = 9$, $\therefore \sqrt{u_B} \geqq 3$, $\therefore x_B, y_B \geqq 3$

$u_C \geqq \mathring{u}_C = 25$, $\therefore \sqrt{u_C} \geqq 5$, $\therefore x_C, y_C \geqq 5$.

$S=\{A,B\}$: S CAN ATTAIN $\sqrt{u_A}+\sqrt{u_B}=10$, FROM (c) OR (a);

$\therefore$ CORE MUST HAVE $\sqrt{u_A}+\sqrt{u_B} \geqq 10$, $x_A + x_B \geqq 10$.

COMBINING THE CONDITIONS FOR $S=N, \{C\}$, AND $\{A,B\}$:

$x_A + x_B + x_C = 15$ i.e., $\sqrt{u_A}+\sqrt{u_B}+\sqrt{u_C}=15$

$x_C \geqq 5$ AND $x_A + x_B \geqq 10$ $\quad$ $\sqrt{u_C} \geqq 5$, $\sqrt{u_A}+\sqrt{u_B} \geqq 10$

$\therefore x_C = 5$, $x_A + x_B = 10$ $\quad$ $\therefore \sqrt{u_C}=5$, $\sqrt{u_A}+\sqrt{u_B}=10$.

$S=\{A,C\}$: $(\mathring{x},\mathring{y})_S = (1,9)+(5,5)=(6,14)$.

UF_S: $\sqrt{u_A}+\sqrt{u_C} = \sqrt{(6)(14)} = \sqrt{84} \cong 9.17$.

CORE REQUIRES $x_i = y_i$, SO $u_A = x_A^2$, $u_C = x_C^2$; $x_A = \sqrt{u_A}$, $x_C = \sqrt{u_C}$

$\therefore x_A + x_C \geqq 9.17$; $\therefore x_A \geqq 4.17$ (SINCE $x_C = 5$); $\therefore x_B \leqq 5.83$.

(NOTE THAT $(4.17)^2 \cong 17.4$, $(5.83)^2 \cong 34.0$.)

$S = \{B, C\}$ IS SYMMETRIC TO $S = \{A, C\}$, SO

$x_B + x_C \geq 9.17$, $\therefore$ $x_B \geq 4.17$, $x_A \leq 5.83$.

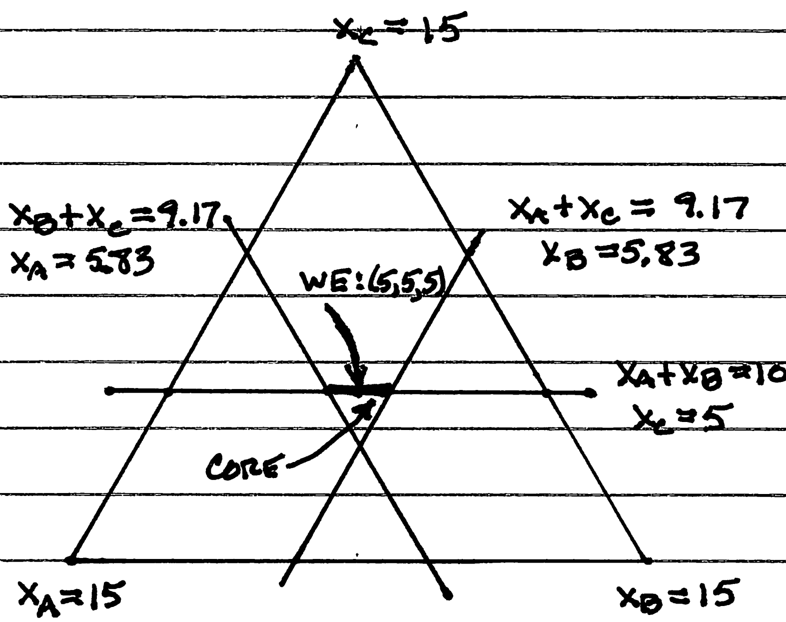
SUMMARIZING:

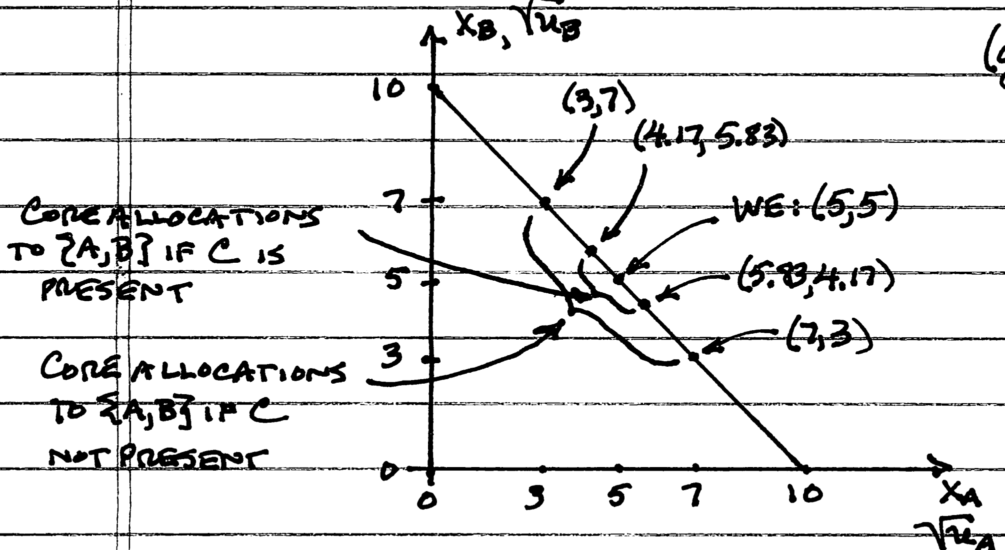
$x_A = y_A$, $x_B = y_B$, $x_C = y_C$

$x_A + x_B + x_C = 15$

$x_C = 5$, $x_A + x_B = 10$

$4.17 \leq x_A \leq 5.83$, $4.17 \leq x_B \leq 5.83$.





(g) A CAN COMBINE WITH C TO IMPROVE ON DEALS SHE CAN'T IMPROVE ON BY HERSELF. (SAME FOR B AND C.) IN EFFECT, A CAN USE C AS A THREAT AGAINST B.

(3) (a) THE INITIAL ALLOCATION IS THE UNIQUE WEA, AT PRICES THAT SATISFY $\frac{P_x}{P_y} = MRS^i(9,16) = \sqrt{\frac{16}{9}} = \frac{4}{3}$.

(b) THE BUNDLE TO BE ALLOCATED, (18, 32), HASN'T CHANGED FROM (a), SO THE PARETO ALLOCATIONS HAVEN'T CHANGED. IN PARTICULAR, WE STILL HAVE $MRS^A = MRS^B = \frac{4}{3}$ AT THAT ALLOCATION, AND $x_A + x_B = \mathring{x}$ AND $y_A + y_B = \mathring{y}$, SO IT IS PARETO OPTIMAL.

THE ALLOCATION IS NOT A WEA FOR THE INITIAL ALLOCATION ((0,32), (18,0)): AT PRICES $(p_x, p_y) = (4,3)$, WE HAVE

$$P_x \bar{x}_A + P_y \bar{y}_A = (4)(0) + (3)(32) = 96$$
$$P_x \bar{x}_B + P_y \bar{y}_B = (4)(18) + (3)(0) = 72$$

ALTERNATIVELY, BUDGET-BALANCE REQUIRES $-\frac{\Delta y}{\Delta x} = \frac{p_x}{p_y}$ i.e., $\frac{p_x}{p_y} = \frac{16}{9} \neq \frac{4}{3}$, SO U-MAX FAILS.

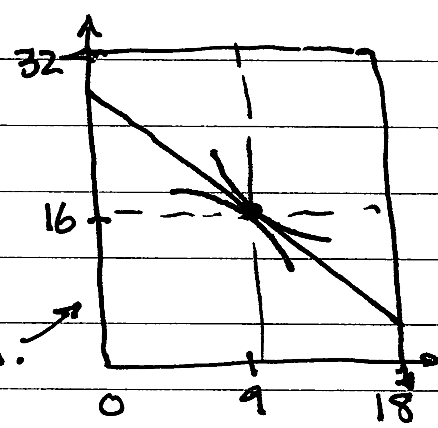
BUT $p_x \mathring{x}_i + p_y \mathring{y}_i = (4)(9) + (3)(16) = 36 + 48 = 84$, $i = A, B$, SO CONSUMERS' BUDGETS DON'T BALANCE (AND THE SAME FOR ANY MULTIPLE OF $(p_x, p_y) = (4,3)$).

FOR ANY OTHER PRICES p_x, p_y WE HAVE $\frac{P_x}{P_y} \neq MRS^i(9,16)$, SO (9,16) DOESN'T MAXIMIZE EITHER CONSUMER'S UTILITY FUNCTION.

(c) THE INITIAL ALLOCATION MUST SATISFY

$$4x_i + 3y_i = (4)(9) + (3)(16)$$
$$= 36 + 48 = 84 \quad (i = A, B)$$

AND $x_A + x_B = 18$, $y_A + y_B = 32$; i.e., MUST LIE ON THE LINE IN THE DIAGRAM.



(d) $q_j = f_j(z_j) = 2z_j \quad (j=1,2)$

$u^i(x_i, y_i) = \sqrt{x_i} + \sqrt{y_i}$, $(\mathring{x}_i, \mathring{y}_i) = (9, 18)$, $MRS^i = \sqrt{\frac{y_i}{x_i}} \quad (i=A,B)$.

THE FIRST WELFARE THEOREM SAYS THAT A W.E. MUST BE PARETO OPTIMAL, SO WE MUST HAVE

$$MRS^i = MRT_j \; [= f_j'(z_j)] = 2 \quad \forall i, j.$$

THEREFORE $\sqrt{\frac{y_i}{x_i}} = 2$, i.e., $\frac{y_i}{x_i} = 4$, i.e., $y_i = 4x_i \quad (i=A,B)$.

TOTAL AMOUNTS x, y, z, q THEREFORE SATISFY

$$y = 4x; \quad z = \mathring{x} - x = 18 - x; \quad q = 2z = 36 - 2x;$$

$$y = \mathring{y} + q = 36 + (36 - 2x) = 72 - 2x.$$

COMBINING $y = 4x$ AND $y = 72 - 2x$:

$$4x = 72 - 2x; \text{ i.e., } 6x = 72; \text{ i.e., } x = 12, \therefore y = 48,$$

$$z = 6, \; q = 12.$$

THE UTILITY-MAXIMIZATION CONDITION FOR EQUILIBRIUM REQUIRES THAT $MRS^i = \frac{p_x}{p_y}$ (INTERIOR), SO WE HAVE $\frac{p_x}{p_y} = 2$. LET $p_x = 2$ AND $p_y = 1$.

CONSUMERS ARE IDENTICAL AND ~~...~~ PROFITS ARE ZERO (PRODUCTION HAS CONSTANT RETURNS TO SCALE); THEREFORE $x_A = x_B = 6$, $y_A = y_B = 24$.

THE INDIVIDUAL FIRMS' PRODUCTION LEVELS ARE INDETERMINATE: ANY z YIELDS ZERO PROFIT. EQUILIBRIUM REQUIRES ONLY THAT THE TWO MARKETS CLEAR: $z_1 + z_2 = 6$, $q_1 = 2z_1$, $q_2 = 2z_2$.

EACH CONSUMER IS ON HER BUDGET CONSTRAINT (AS (U-MAX) REQUIRES):

$$p_x x_i + p_y y_i = (2)(6) + (1)(24) = 36; \; p_x \mathring{x}_i + p_y \mathring{y}_i = (2)(9) + (1)(18) = 36.$$