Economics 501B Final Exam Fall 2017
Solutions

1. For each of the following propositions, state whether the proposition is true or false. If true, provide a proof (or at least indicate how a proof could be constructed). If false, provide a counterexample and verify that it is a valid counterexample.

(a) If each consumer's utility function is

(a1) continuous,

(a2) quasiconcave (*i.e.*, upper-contour sets are convex), and

(a3) weakly increasing (*i.e.*, if $\widetilde{x}_k \geqq x_k$ for each good k, then $u(\widetilde{\mathbf{x}}) \geqq u(\mathbf{x})$),

then any Walrasian equilibrium allocation is Pareto optimal.

(b) If every consumer has a lexicographic preference, then there is no Walrasian equilibrium.

Solution: Both propositions are false. Here are counterexamples to each proposition:

(a) There are two consumers, one of whom has a thick indifference curve:

$$u^A(x,y) = \begin{cases} xy, & \text{if } xy \leqq 16 \\ 16, & \text{if } 16 \leqq xy \leqq 36 \\ xy - 20, & \text{if } xy \geqq 36 \end{cases} \qquad \text{and} \qquad u^B(x,y) = xy.$$

Each consumer's endowment is $(\mathring{x}_i, \mathring{y}_i) = (5,5)$. The unique Walrasian equilibrium is $p_x = p_y$ with no trade: each consumer chooses to consume her initial bundle, $(5,5)$. This allocation is not Pareto optimal: the allocation $\big((x_A, y_A), (x_B, y_B)\big) = \big((4,4),(6,6)\big)$ is a Pareto improvement, because it yields $(u_A, u_B) = (16, 36)$ while $(\mathring{u}_A, \mathring{u}_B) = (16, 25)$.

(b) There are two consumers and two goods. Consumer A prefers the x-good and Consumer B prefers the y-good (see Figure 1):

$$(x', y') \succsim_A (x,y) \Leftrightarrow \big[x' > x \text{ or } [x' = x \;\&\; y' \geqq y]\big]$$
$$(x', y') \succsim_B (x,y) \Leftrightarrow \big[y' > y \text{ or } [y' = y \;\&\; x' \geqq x]\big]$$

The equilibrium prices satisfy $p_x/p_y = \mathring{y}_A/\mathring{x}_B$. Consumer A will sell all his y-good to buy the x-good, and Consumer B will sell all his x-good to buy the y-good, so the equilibrium allocation is $\big((x_A, y_A), (x_B, y_B)\big) = \big((\mathring{x}, 0), (0, \mathring{y})\big)$. Note that this is on each consumer's budget constraint.

2. When we have a parametric family of optimization problems $P(\theta)$ for parameter values θ in some set Θ of possible parameter values, we're usually interested in the solution function (or the solution correspondence) for the set $\{P(\theta) \,|\, \theta \in \Theta\}$, and we're often interested in the value function as well. An application of this idea arises in the concept of Pareto optimality. The simplest case is a "two by two exchange economy," where there are only two goods and only two consumers, production is not possible and there are no externalities — and where we use the Edgeworth box diagram to graphically depict some of our economic concepts. Assume that both consumers' preferences are representable by continuous, strictly increasing, strictly concave utility functions.

Give a careful explanation of how the following concepts are related to one another in such an "Edgeworth box" economy: the optimization problems $P(\theta)$ and what economic parameters are playing the role of theta; the solution function or correspondence; the Pareto allocations; the graph of the Pareto allocations in the box; the value function; and the utility frontier (also called the Pareto frontier). Why do we refer to the value function and never to the value correspondence?

Solution: The Pareto allocations in this economy are the solutions of the P-Max problem $P(u_2)$:

$$\max_{\mathbf{x}\in\mathbb{R}^4_+} u^1(x^1_1, x^1_2) \text{ subject to}$$
$$x^1_1 + x^2_1 \leqq \mathring{x}_1$$
$$x^1_2 + x^2_2 \leqq \mathring{x}_2$$
$$u^2(x^2_1, x^2_2) \geqq u_2 \ ,$$

for the various values of $u_2 \in \mathbb{R}$.

The parameter that plays the role of θ is u_2, the target utility level for Consumer 2. For each value of u_2 (*i.e.*, each value of θ), the solution $\widehat{\mathbf{x}} = \big((\widehat{x}^1_1, \widehat{x}^1_2), (\widehat{x}^2_1, \widehat{x}^2_2)\big)$ of the problem $P(u_2)$ is the Pareto allocation in which Consumer 2's utility level is u_2. Therefore the solution function $\widehat{\mathbf{x}}(u_2)$ tells us the Pareto allocation as a function of u_2, and the range of the solution function is the set of all Pareto allocations, which is the locus of indifference-curve tangencies in the Edgeworth box. (The solution function is indeed a function — *i.e.*, a singleton-valued correspondence — because each utility function is strictly concave.) The value function $v(\theta)$ (*i.e.*, $v(u_2)$) tells us Person 1's utility level u_1 at the Pareto allocation in which Person 2's utility level is u_2, so the graph of the value function $u_1 = v(u_2)$ is the utility frontier.

The value function has to be single-valued: if for a given θ there were two solutions that gave different values of the objective function, then the solution with the smaller value would not actually be a solution.

3. Amy and Bev both have flower gardens. Their gardens are positioned in such a way that Amy can see Bev's garden as well as her own, and Amy therefore derives "utility" both from Bev's garden and her own garden. But Bev can't see Amy's garden, so she derives utility only from her own garden. Their preferences are represented by the utility functions

$$u^A(x_A, y_A, x_B) = y_A + 12x_A - \tfrac{1}{2}x_A^2 + 6x_B - \tfrac{1}{2}x_B^2 \text{ and}$$
$$u^B(x_B, y_B) = y_B + 8x_B - \tfrac{1}{2}x_B^2 \text{ ,}$$

where y_i is i's consumption of dollars and x_i is the size of i's garden, in square meters. The cost of a garden is four dollars per square meter, and each woman is endowed with 100 dollars.

(a) Write down a maximization problem for which the solutions are the Pareto allocations.

(b) Derive the first-order conditions that characterize the solution(s) of the problem in (a).

(c) Determine the Pareto optimal allocations.

(d) Determine the utility (*i.e.*, Pareto) frontier.

(e) Express the first-order conditions in terms of marginal rates of substitution, and suggest prices and per-unit taxes or subsidies that would yield a Pareto allocation as an equilbrium if Amy and Bev are both price-takers, even if there is no way for Amy to purchase flowers for Bev's garden.

Solution:

(a) $\max_{\mathbf{x}\in\mathbb{R}^4_+} u^A(x_A, y_A, x_B)$ subject to $u^B(x_B, y_B) \geqq u_B$ and $4x_A + 4x_B + y_A + y_B \leqq \mathring{y} = 200$.

(b) First-order marginal conditions: $\exists\, \lambda_B, \sigma \geqq 0$ such that

$$\begin{aligned} x_A :&\quad 12 - x_A \leqq 4\sigma \\ y_A :&\quad 1 \leqq \sigma \\ x_B :&\quad 6 - x_B \leqq 4\sigma - \lambda_B(8 - x_B) \\ y_B :&\quad 0 \leqq \sigma - \lambda_B \end{aligned}$$

and in each case the inequality is an equation if the corresponding variable is positive. The remaining conditions are that the two constraint inequalities are satisfied, and that the first is an equation if $\lambda_B > 0$ and the second is an equation if $\sigma > 0$.

(c) If all four variables are positive we have

$\lambda_B = \sigma = 1$

$12 - x_A = 4$, therefore $\boxed{x_A = 8}$

$(6 - x_B) + (8 - x_B) = 4$, therefore $2x_B = 14 - 4 = 10$, *i.e.*, $\boxed{x_B = 5}$.

$\boxed{y_A + y_B = 200 - (4)(13) = 148}$.

In fact, since $y_A, y_B > 0 \implies \lambda_B = \sigma = 1$, this ensures that $x_A \geqq 12-4 > 0$ and $2x_B \geqq 14-4 > 0$, so these are all the Pareto allocations in which $y_A, y_B > 0$. If either $y_A = 0$ or $y_B = 0$, see below.

(d) At the Pareto allocations in (c) we have

$$u_A = y_A + 12x_A - \tfrac{1}{2}x_A^2 + 6x_B - \tfrac{1}{2}x_B^2 = y_A + 96 - 32 + 30 - 12.5 = y_A + 81.5 \quad \text{and}$$
$$u_B = y_B + 8x_B - \tfrac{1}{2}x_B^2 = y_B + 40 - 12.5 = y_B + 27.5.$$

Therefore $u_A + u_B = y_A + y_B + 109$.

The cost of $x_A + x_B = 8 + 5 = 13$ is $(4)(13) = 52$, so $y_A + y_B = 200 - 52 = 148$, so we have $u_A + u_B = 148 + 109 = 257$.

So the utility frontier is $\boxed{u_A + u_B = 257.}$

But that's for Pareto allocations in which $y_A, y_B > 0$. What if either $y_A = 0$ or $y_B = 0$? Here things get slightly more complicated — I hadn't noticed this issue when I wrote this question, unfortunately.

First notice that if either $y_A = 0$ or $y_B = 0$, the allocation in (c) is still a Pareto allocation, but now it's not the *only* Pareto allocation. Next notice that to have $y_A \geqq 0$ on the utility frontier $u_A + u_B = 257$ is to have $u_A \geqq 81.5$ and therefore $u_B \leqq 175.5$. And to have $y_B \geqq 0$ is to have $u_B \geqq 27.5$, and therefore $u_A \leqq 229.5$. So between the points $(u_A, u_B) = (81.5, 175.5)$ and $(u_A, u_B) = (229.5, 27.5)$ the utility frontier is as above, $u_A + u_B = 257$. See Figure 2.

If you simply obtained $u_A + u_B = 257$ for the utility frontier, that's fine. For the remainder of the utility frontier, and the remaining Pareto allocations, see the appendix I've added at the end of these solutions.

(e) Note that $MRS^A = 12 - x_A$, $MRS^B = 8 - x_B$, and $MRS^A_B = 6 - x_B$, where MRS^A_B denotes the marginal rate of substitution of Consumer A for the good x_B. So the first-order marginal conditions at interior allocations, from (c), are $MRS^A = MC$ and $MRS^B + MRS^A_B = MC$. $MC = 4$, so we have $12 - x_A = 4$ and $(8 - x_B) + (6 - x_B) = 4$, so the interior Pareto allocations are at $x_A = 8$ and $x_B = 5$. Note that at $x_B = 5$ we have $MRS^A_B = 1$.

If Amy can't affect x_B directly, the price system could subsidize Bev by an amount $s = \$1$ per unit. This would lower her net price from \$4 per unit to \$3 per unit, leading her to choose x_B to satisfy $MRS^B = p - \$1$, *i.e.*, $8 - x_B = 3$, *i.e.*, $x_B = 5$ (assuming that $p = MC = \$4$). Of course this depends critically on Bev being a price-taker, both for the price per unit and the subsidy per unit. While this is unrealistic when there are only two people in the market, it's a more reasonable assumption when there are "enough" people in the market.

4. Two Manhattan pretzel vendors must decide where to locate their pretzel carts along a given block of Fifth Avenue, represented by the unit interval $I = [0,1] \subseteq \mathbb{R}$ — *i.e.*, each vendor chooses a location $x_i \in [0,1]$. The profit of each vendor i depends continuously on *both* vendors' locations — *i.e.*, the profit functions $\pi_i : I \times I \to \mathbb{R}$ are continuous for $i = 1,2$. Furthermore, each π_i is concave (but not strictly concave) in x_i.

Define an equilibrium in this situation to be a joint action $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in I^2$ that satisfies both

$$\forall x_1 \in I : \pi_1(\widehat{x}) \geqq \pi_1(x_1, \widehat{x}_2) \qquad \text{and} \qquad \forall x_2 \in I : \pi_2(\widehat{x}) \geqq \pi_2(\widehat{x}_1, x_2).$$

In other words, an equilibrium consists of a location for each vendor, with the property that each one's location is best for him given the other's location.

Prove that an equilibrium exists. If you're unable to prove this for the case in which each π_i is merely concave, assume they're both *strictly* concave in x_i and prove the result in that case.

Solution:

We'll show that Vendor 1's reaction correspondence $\mu_1 : I \twoheadrightarrow I$ is nonempty-valued, convex-valued, and closed. The same argument will apply to Vendor 2's reaction correspondence μ_2. Let $\varphi : I \twoheadrightarrow I$ be the feasible-set correspondence $\varphi(x_2) = I$, which is constant and therefore continuous, and is also nonempty-, compact-, and convex-valued. Let $\mu_1 : I \to I$ be the reaction correspondence, $\mu_1(x_2) = \text{argmax}_{\varphi(x_2)} \pi_1(x_1, x_2)$. Since π_1 is continuous, all the conditions of the Maximum Theorem are satisfied, therefore μ_1 is a closed correspondence. Moreover, for each $x_2 \in I$, the set $\mu_1(x_2)$ of maximizers of $\pi_1(x_1, x_2)$ is a convex set because π_1 is concave in the variable x_1 (this was Problem #3 on the Econ 519 exam), and is compact because $\mu_1(x_2)$ is closed and the target space I is bounded, so that $\mu_1(x_2)$ is bounded.

Now define a "transition correspondence" $f : I \times I \twoheadrightarrow I \times I$ as follows:

$$\forall (x_1, x_2) \in I^2 : f(x_1, x_2) = \mu_1(x_2) \times \mu_2(x_1).$$

The set $I^2 = I \times I$ is nonempty, compact, and convex, and the correspondence f is nonempty-valued, convex-valued (each set $f(x_1, x_2)$ is the product of two convex sets), and closed (as the Cartesian product of closed correspondences). Therefore the Kakutani Fixed Point Theorem ensures that f has a fixed point, $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2)$ — a point at which $\widehat{x}_1 \in \mu_1(\widehat{x}_2)$ and $\widehat{x}_2 \in \mu_2(\widehat{x}_1)$, *i.e.*, a Nash equilibrium. ■

If you assumed that each π_i is *strictly* concave, see the solution to Problem #5 on the Econ 519 final exam.

5. In a two-period, one-good model where S is the set of possible states, one of which will occur after period zero and prior to period one, let $\mathbf{p} \in \mathbb{R}^S$ be a state-contingent price-list; let D be an $S \times K$ securities-returns matrix; and let $\mathbf{q} = \mathbf{p}D \in \mathbb{R}^K$. The following proposition appears in our lecture notes (where S denotes the number of states as well as the set of states):

Proposition: Let

$$A = \{(z_0, \mathbf{z}) \in \mathbb{R}^{1+S} \mid z_0 + \mathbf{p} \cdot \mathbf{z} = 0\} \text{ and}$$
$$B = \{(z_0, \mathbf{z}) \in \mathbb{R}^{1+S} \mid \exists \mathbf{y} \in \mathbb{R}^K : z_0 + \mathbf{q} \cdot \mathbf{y} = 0 \text{ and } \mathbf{z} = D\mathbf{y}\}.$$

If $rank\, D = S$, then $A = B$.

(a) The $(1+S)$-tuples $(z_0, \mathbf{z})$ represent net consumption bundles and the K-tuples $\mathbf{y}$ represent holdings of securities. Describe what the proposition tells us in economic terms, and describe the role the proposition plays in establishing the relation between an Arrow-Debreu contingent-claims-markets equilibrium and an equilibrium in Arrow's model of securities and spot markets.

(b) Provide a proof of the proposition.

Solution:

(a) The sets A and B are the sets of net consumption bundles ("net trades") a consumer can obtain in either the Arrow-Debreu market structure (the set A) or the Arrow securities market structure (the set B). If the two sets are equal (*i.e.*, the same set), then the consumption bundles available to the consumer are exactly the same in the two situations, so she will choose the same bundle in each situation. Therefore, since each consumer's choice of consumption bundle will be the same in A as in B, the prices $\mathbf{p}$ will be equilibrium prices in the A-D markets if and only if $\mathbf{q} = \mathbf{p}D$ are equilibrium prices in the Arrow markets, and the equilibrium consumptions will be the same in each market structure. The condition that's sufficient to guarantee this, that $rank\, D = S$, is that among the K securities there are S linearly independent securities — *i.e.*, S securities whose returns vectors are linearly independent and therefore span the space $\mathbb{R}^S$.

(b) Note that if $\mathbf{z} = D\mathbf{y}$ then $\mathbf{p} \cdot \mathbf{z} = \mathbf{p} \cdot (D\mathbf{y}) = (\mathbf{p}D) \cdot \mathbf{y} = \mathbf{q} \cdot \mathbf{y}$. We show that $A \subseteq B$ and $B \subseteq A$.

(i) Let $(z_0, \mathbf{z}) \in A$. Since $rank\, D = S$, there is a $\mathbf{y} \in \mathbb{R}^K$ that satisfies $\mathbf{z} = D\mathbf{y}$. Since $z_0 + \mathbf{p} \cdot \mathbf{z} = 0$ (because $(z_0, \mathbf{z}) \in A$) and $\mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot \mathbf{y}$ (because $\mathbf{z} = D\mathbf{y}$), we have $z_0 + \mathbf{q} \cdot \mathbf{y} = 0$, and therefore $(z_0, \mathbf{z}) \in B$.

(ii) Let $(z_0, \mathbf{z}) \in B$. Then, according to the definition of B, there is a $\mathbf{y} \in \mathbb{R}^K$ that satisfies both $z_0 + \mathbf{q} \cdot \mathbf{y} = 0$ and $\mathbf{z} = D\mathbf{y}$. Therefore $\mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot \mathbf{y}$, and it follows that $z_0 + \mathbf{p} \cdot \mathbf{z} = 0$, and therefore $(z_0, \mathbf{z}) \in A$. ■

Appendix for Problem #3

We want to determine the Pareto allocations in which either $y_A = 0$ or $y_B = 0$.

First consider the case $y_A > 0$ and $y_B = 0$. In this case it will be Pareto optimal to simply maximize $u^A(\cdot)$ subject to the constraint $4x_A + 4x_B + y_A = 200$, because there is no way to transfer any dollars (the y-good) from B to A to compensate A for accepting any other allocation (because $y_B = 0$). Therefore we will have $x_A = 8$ and $x_B = 2$, the cost of which is $(4)(10) = 40$, so we have $y_A = 200 - 40 = 160$. This yields

$u^A(8, 160, 2) = 160 + (12)(8) - \frac{1}{2}(8)^2 + (6)(2) - \frac{1}{2}(2)^2 = 160 + 96 - 32 + 12 - 2 = 234$
$u_B(2, 0) = 0 + (8)(2) - \frac{1}{2}(2)^2 = 16 - 2 = 14.$

The utility frontier therefore includes the additional point $(u_A, u_B) = (234, 14)$. See Figure 2.

Note that the FOMC are satisfied with $\sigma = 1$ and $\lambda_B = 0$, so this is also the solution of our P-Max problem for any target $u_B \leqq 14$: the utility-target-level constraint is not binding at the solution, at which $u_B = 14$.

Now consider the case $y_A = 0$ and $y_B > 0$. In this case it will be Pareto optimal to simply maximize $u^B(\cdot)$ subject to the constraint $4x_B + y_B = 200$, because there is no way to transfer any dollars (the y-good) from A to B to compensate B for accepting any other allocation (because $y_A = 0$). Therefore we will have $x_A = 0$ and $x_B = 4$, the cost of which is $(4)(4) = 16$, so we have $y_B = 200 - 16 = 184$. This yields

$u^A(0, 0, 4) = 0 + (6)(4) - \frac{1}{2}(4)^2 = 24 - 8 = 16$
$u_B(4, 184) = 184 + (8)(4) - \frac{1}{2}(4)^2 = 184 + 32 - 8 = 208.$

The utility frontier therefore includes the additional point $(u_A, u_B) = (16, 208)$. This is also depicted in Figure 2.

Note that in this case the FOMC are *not* satisfied at this solution. But we know that any Pareto allocation must be the solution of the P-Max problem in which the utility-level-target is the value of u_B at the solution. What's going wrong?

The answer is that the constraint set is a singleton, a constraint-qualification violation, and therefore this is a case in which the maximum need not satisfy the Kuhn-Tucker Conditions.

However, the allocation must also be the solution of the P-Max problem in which we maximize $u^B(\cdot)$ subject to a utility-level constraint on u^A. If you do this, you'll find that the FOMC are satisfied with $\sigma = 1$ and $\lambda_A = 0$, for any $u_A \leqq 16$.

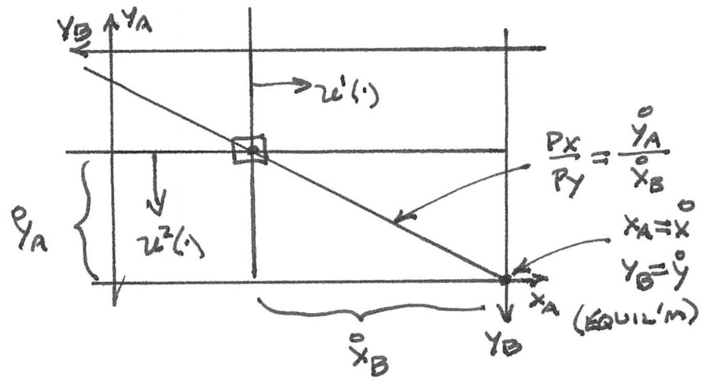

Figure 1: Edgeworth Box for #1(b)

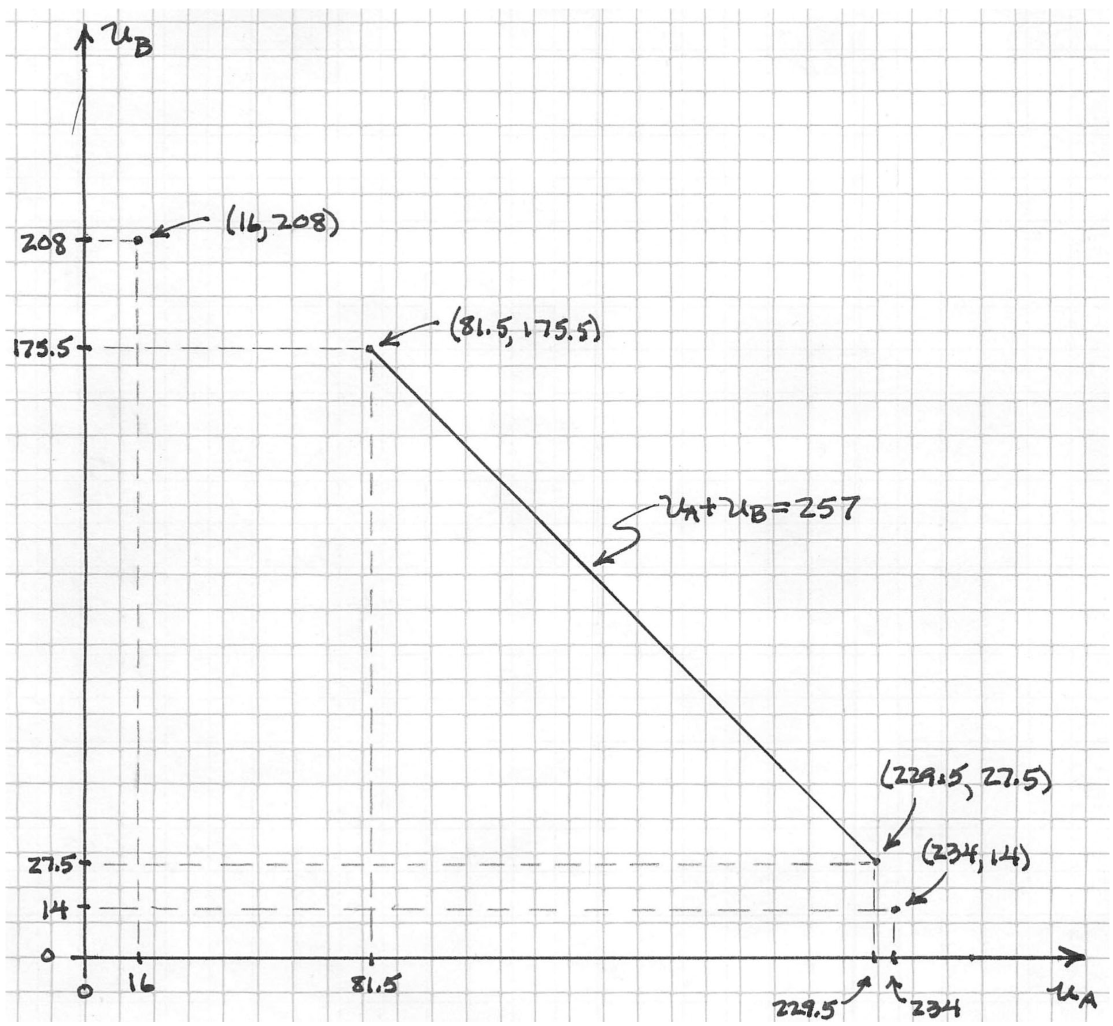

Figure 2: Utility Frontier for #3(d)