Economics 501B Final Exam Solutions
Fall 2016

1. Two firms (Firm 1 and Firm 2) each sell spring water, directly from the source: no matter how much they sell, the cost to them is zero. The market demand functions for their water are

$$q_1 = 60 - 2p_1 + p_2 \qquad \text{and} \qquad q_2 = 60 + p_1 - 2p_2,$$

where q_i denotes the number of gallons Firm i sells and p_i denotes the price (in dollars) Firm i charges for each gallon. Each firm chooses its price to maximize its profit (which, because costs are zero, is equivalent to its revenue).

(a) Assume that each firm takes its rival's price as given, and determine their Bertrand reaction functions, draw the reaction functions in a diagram, and determine the Bertrand equilibrium prices, quantities, and profits (revenues).

Solution:

The firms' profit/revenue functions are

$$R_1(p_1, p_2) = (60 + p_2)p_1 - 2p_1^2 \qquad \text{and} \qquad R_2(p_1, p_2) = (60 + p_1)p_2 - 2p_2^2$$

The firms' first-order marginal conditions and reaction functions (see Figure 1) are

$$\frac{\partial R_1}{\partial p_1} = 60 + p_2 - 4p_1 = 0 \iff 4p_1 - p_2 = 60; \quad i.e.,\ p_1 = 15 + \tfrac{1}{4}p_2,$$
$$\frac{\partial R_2}{\partial p_2} = 60 + p_1 - 4p_2 = 0 \iff 4p_2 - p_1 = 60; \quad i.e.,\ p_2 = 15 + \tfrac{1}{4}p_1.$$

The Bertrand equilibrium: $\quad p_1 = p_2 = \$20; \quad q_1 = q_2 = 40; \quad \pi_1 = \pi_2 = R_1 = R_3 = \$800.$

(b) Are the two firms' products identical, or are they differentiated from one another? Explain how you can tell whether they're identical or differentiated.

Solution:

If buyers are unable to distinguish one firm's product from the other firm's product, then the firm charging the lower price will garner all or nearly all the sales. However, the demand functions here allow for the firms to charge significantly different prices without the low-price firm capturing nearly all the market. (Even otherwise identical products can be differentiated by such features as location, branding, etc.)

For the remainder of this problem it may be helpful to know that the following two matrices are inverses of one another, as you can easily check:

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(c) Now suppose that instead of taking its rival's price as given, each firm takes its rival's output as given. Determine the Cournot reaction functions, draw the reaction functions in a diagram, and determine the Cournot equilibrium quantities, prices, and profits (revenues).

Solution:

The demand functions can be inverted, so that the market-clearing prices are expressed in terms of the quantities the firms offer to sell (their inverse demand functions):

$$\mathbf{q} = \mathbf{b} - A\mathbf{p}, \qquad i.e., \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

when inverted is

$$\mathbf{p} = A^{-1}(\mathbf{b} - \mathbf{q}) = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 60 - q_1 \\ 60 - q_2 \end{bmatrix} = \begin{bmatrix} 60 - \frac{2}{3}q_1 - \frac{1}{3}q_2 \\ 60 - \frac{1}{3}q_1 - \frac{2}{3}q_2 \end{bmatrix}.$$

The firms' profit/revenue functions are

$$\widetilde{R}_1(q_1, q_2) = (60 - \tfrac{1}{3}q_2)q_1 - \tfrac{2}{3}q_1^2 \quad \text{and} \quad \widetilde{R}_2(q_1, q_2) = (60 - \tfrac{1}{3}q_1)q_2 - \tfrac{2}{3}q_2^2.$$

The firms' first-order marginal conditions and reaction functions (see Figure 2) are

$$\frac{\partial \widetilde{R}_1}{\partial q_1} = 60 - \tfrac{1}{3}q_2 - \tfrac{4}{3}q_1 = 0 \iff \tfrac{4}{3}q_1 + \tfrac{1}{3}q_2 = 60; \quad i.e., \; q_1 = 45 - \tfrac{1}{4}q_2,$$
$$\frac{\partial \widetilde{R}_2}{\partial q_2} = 60 - \tfrac{1}{3}q_1 - \tfrac{4}{3}q_2 = 0 \iff \tfrac{1}{3}q_1 + \tfrac{4}{3}q_2 = 60; \quad i.e., \; q_2 = 45 - \tfrac{1}{4}q_1.$$

(d) Now suppose that Firm 2 is charging $p_2 = \$24$ per gallon and is selling $q_2 = 36$ gallons. Assume that Firm 1 takes the quantity $q_2 = 36$ as given. Determine Firm 1's residual demand function, and draw its residual demand curve and its marginal revenue curve in a single diagram. Depict Firm 1's profit-maximizing decision in the diagram.

Solution:

Firm 1's residual demand function (see Figure 3) is

$$
\begin{aligned}
p_1 &= 60 - \tfrac{2}{3}q_1 - \tfrac{1}{3}q_2 \\
&= 60 - \tfrac{2}{3}q_1 - 12 \\
&= 48 - \tfrac{2}{3}q_1.
\end{aligned}
$$

Therefore Firm 1's marginal revenue function is $MR_1 = 48 - \frac{4}{3}q_1$, and $MR_1 = MC$ at $q_1 = 36$. Therefore $p_1 = \$24$. Note that if Firm 2 also takes Firm 1's quantity to be fixed at $q_1 = 36$, then this is an equilibrium, the Cournot equilibrium.

(e) Now assume that Firm 1 instead takes Firm 2's price $p_2 = \$24$ as given. Determine Firm 1's residual demand function, and draw its residual demand curve and its marginal revenue curve in a single diagram. Depict Firm 1's profit-maximizing decision in the diagram.

Solution:

Firm 1's residual demand function (see Figure 4) is

$$
\begin{aligned}
q_1 &= 60 - 2p_1 + p_2 \\
&= 60 - 2p_1 + 24 \\
&= 84 - 2p_1,
\end{aligned}
$$

i.e.,

$$p_1 = 42 - \tfrac{1}{2}q_1.$$

Firm 1's marginal revenue function is therefore $MR_1 = 42 - q_1$, and $MR_1 = MC$ at $q_1 = 42$. Therefore $p_1 = \$21$. This is *not* consistent with an equilibrium in which Firm 2 takes either Firm 1's price to be \$21 or its quantity to be 42: in either case Firm 2 will not choose $p_2 = \$24$ and $q_2 = 36$.

See also Figure 5.

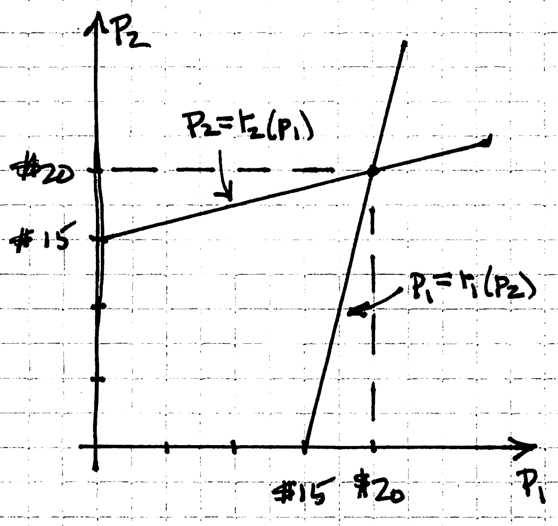

FIGURE 1

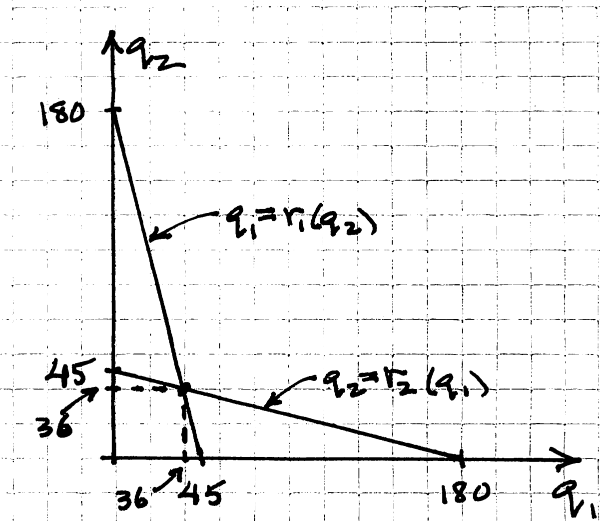

FIGURE 2

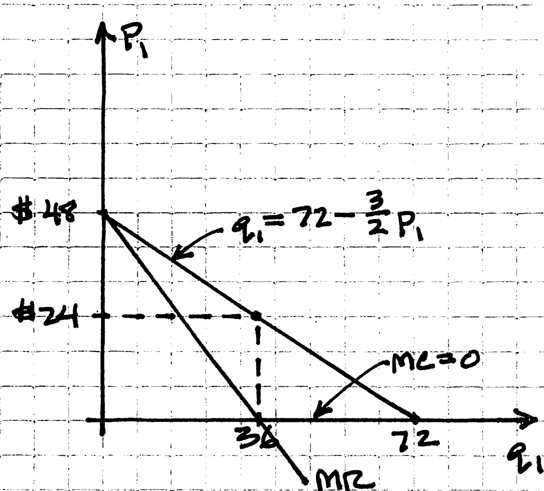

FIGURE 3

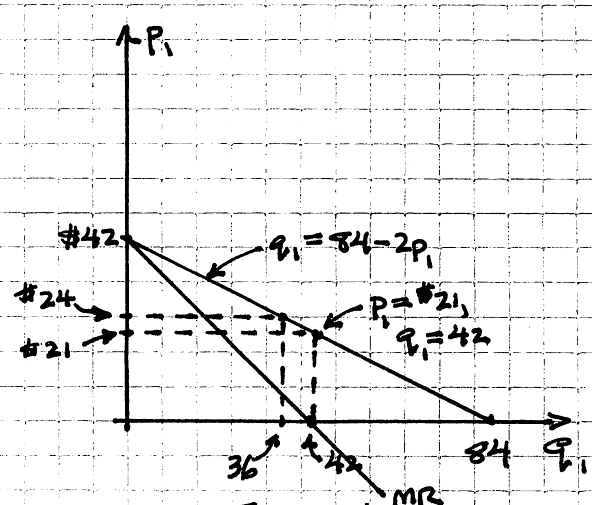

FIGURE 4

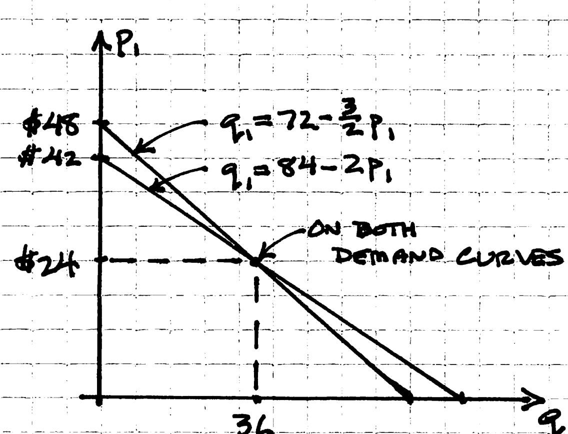

FIGURE 5

2. Suppose we have a model of the economy in which there are only two periods, $t = 0$ and $t = 1$ ("today" and "tomorrow"), and in which there is only a single good, which we'll call simoleons (or you can call it dollars if you like). Consumers are endowed with some units of the good today, and will be endowed again with some units tomorrow, but their endowments tomorrow will depend (in a way that's known today) on which one of three states of the world will have occurred after today and before tomorrow: $s = H$, or $s = M$, or $s = L$. Let $S = \{H, M, L\}$. Consumers have differing state-dependent preferences over the space $\mathbb{R}^4_+$ of consumption bundles $\mathbf{x}^i = (x^i_0, x^i_H, x^i_M, x^i_L)$. No production or storage is possible. Assume that the unique Arrow-Debreu complete contingent claims equilibrium prices are

$$p_H = \frac{1}{8}, \quad p_M = \frac{1}{4}, \quad p_L = \frac{1}{4}.$$

In (a), (b), and (c), below, you're given three alternative sets of securities. The three components of a security d_k are the number d_{sk} of simoleons that a unit of the security will pay to the holder tomorrow in each of the three states; *i.e.*,

$$d_k = \begin{bmatrix} d_{kH} \\ d_{kM} \\ d_{kL} \end{bmatrix}.$$

In (a), (b), and (c) determine each of the following, if it's possible; if it's not, explain why not:

- The equilibrium prices ψ_k of each of the securities.
- The equilibrium interest rate.
- How many units y_k of each of the securities a consumer would need to hold in order to ensure that she will receive the state-dependent payout $(z_H, z_M, z_L) = (1, 2, 2)$.
- How much it will cost her today to ensure she will receive $(1, 2, 2)$ tomorrow.
- If for one of the securities market structures there is more than one list $\mathbf{y}$ of holdings y_k that will achieve the payout $(1, 2, 2)$, indicate one of the additional vectors $\mathbf{y}$ that will attain $(1, 2, 2)$ and determine how much that will cost the consumer today.

(a) $d_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $d_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) $d_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, $d_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $d_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

(c) $d_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $d_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $d_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $d_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution:

(a) There are only two securities and there are three states, therefore the securities do not span the space of possible returns, $\mathbb{R}^3$. Therefore there is no determinate relation between the Arrow-Debreu prices and the equilibrium security prices ψ_1 and ψ_2. For the same reason, we don't have enough information to determine the equilibrium interest rate. Despite the fact that the securities don't span $\mathbb{R}^3$, it's possible to obtain the vector $(1, 2, 2)$ of returns, by holding one unit of each security: $(1, 2, 2) = d_1 + d_2$. Without knowing the prices ψ_k we can't determine how much this will cost the consumer.

(b) The securities are linearly independent: if $\lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 = (0, 0, 0)$, then clearly $\lambda_1 = 0$ (in order to obtain 0 for the third component); and therefore it's clear that $\lambda_2 = 0$ (to obtain 0 for the second component); and therefore we must have $\lambda_3 = 0$ (to obtain 0 for the first component). Therefore the securities' prices can be determined from the Arrow-Debreu prices:

$$\psi_1 = 4p_H + 4p_M + 4p_L = 4\left(\tfrac{1}{8} + \tfrac{1}{4} + \tfrac{1}{4}\right) = (4)\left(\tfrac{5}{8}\right) = \tfrac{20}{8} = 2\tfrac{1}{2}$$

$$\psi_2 = p_H = \tfrac{1}{8} \quad \text{and} \quad \psi_3 = 2p_H + 2p_M = (2)\left(\tfrac{1}{8}\right) + (2)\left(\tfrac{1}{4}\right) = \tfrac{3}{4}.$$

The equilibrium interest rate r satisfies

$$\frac{1}{(1+r)} = p_H + p_M + p_L = \tfrac{5}{8};$$

therefore $r = \frac{3}{5} = 60\%$. In order to achieve the returns vector $(1, 2, 2)$, the consumer will have to hold the portfolio $(y_1, y_2, y_3) = (\frac{1}{2}, -1, 0)$. Because the securities are linearly independent, this is the only vector $\mathbf{y}$ that will achieve $(1, 2, 2)$. The net cost is $(\frac{1}{2})(2\frac{1}{2}) + (-1)(\frac{1}{8}) = \frac{5}{4} - \frac{1}{8} = \frac{9}{8}$.

(c) The securities span $\mathbb{R}^3$ — in fact, it's obvious that the securities d_2, d_3, and d_4 span $\mathbb{R}^3$ — so the equilibrium security prices can be determined from the Arrow-Debreu prices:

$$\psi_1 = p_H + p_M + p_L = \tfrac{5}{8}, \quad \psi_2 = p_H = \tfrac{1}{8}, \quad \psi_3 = p_M = \tfrac{1}{4}, \quad \psi_4 = p_L = \tfrac{1}{4}.$$

The equilibrium interest rate is again $r = \frac{3}{5}$, as in (b). The holdings vector $\mathbf{y} = (0, 1, 2, 2)$ will clearly yield the returns vector $(1, 2, 2)$, and it will cost

$$\psi_2 + 2\psi_3 + 2\psi_4 = \tfrac{1}{8} + (2)\left(\tfrac{1}{4}\right) + (2)\left(\tfrac{1}{4}\right) = \tfrac{9}{8}.$$

Because the securities are not linearly independent but do span $\mathbb{R}^3$, there are other holdings, besides the $\mathbf{y}$ above, that will yield the returns vector $(1, 2, 2)$ — for example, $\mathbf{y} = (1, 0, 1, 1)$, which will cost

$$\psi_1 + \psi_3 + \psi_4 = \tfrac{5}{8} + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{9}{8}.$$

3. Suppose there are two consumers and that $\big(\widehat{\mathbf{p}}, (\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)\big) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^{2\ell}_+$ is a Walrasian equilibrium for preferences $\succsim^1$ and $\succsim^2$ on $\mathbb{R}^\ell_+$ and endowments $\mathring{\mathbf{x}}^1$ and $\mathring{\mathbf{x}}^2$ in $\mathbb{R}^\ell_+$. Assuming only that each preference is a complete and locally nonsatiated (LNS) preorder of $\mathbb{R}^\ell_+$, prove that the allocation $(\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)$ is in the core. (Note that the preferences might not be differentiable, or continuous, or quasiconcave, etc., and they might not be representable by utility functions.) Before giving your proof, give the definition of a Walrasian equilibrium and the definition of the core for this two-consumer economy. If you find it easier to assume there are only two goods, it's OK to assume that.

Solution:

Definition: A Walrasian equilibrium of the economy $\big((\succsim^1, \mathring{\mathbf{x}}^1), (\succsim^2, \mathring{\mathbf{x}}^2)\big)$ is a pair $\big(\widehat{\mathbf{p}}, (\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)\big) \in \mathbb{R}^\ell_+ \times \mathbb{R}^{2\ell}_+$ that satisfies

(U-Max) For each i:

$\widehat{\mathbf{x}}^i$ is maximal according to $\succsim^i$ in the budget set $\{\mathbf{x} \in \mathbb{R}^\ell_+ \mid \widehat{\mathbf{p}} \cdot \mathbf{x} \leqq \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i\}$, and

(M-Clr) For $k = 1, \ldots, \ell:$ $x^1_k + x^2_k \leqq \mathring{x}^1_k + \mathring{x}^2_k$ and $(x^1_k + x^2_k) = (\mathring{x}^1_k + \mathring{x}^2_k)$ if $p_k > 0$.

Definition: The core of the economy $\big(\succsim^1, \succsim^2, \mathring{\mathbf{x}})\big)$ is the set of feasible allocations $(\mathbf{x}^1, \mathbf{x}^2)$ that no coalition can unilaterally improve upon.

Proof of the proposition:

We first prove that no one-person coalition (*i.e.*, no individual) can unilaterally improve upon the allocation $(\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)$: Suppose that $\widetilde{\mathbf{x}}^i \succ^i \widehat{\mathbf{x}}^i$; then by (U-Max) we have $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i > \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i$. Since $\widehat{\mathbf{p}} \in \mathbb{R}^\ell_{++}$, we must have either $\widetilde{\mathbf{x}}^i_1 > \mathring{\mathbf{x}}^i_1$ or $\widetilde{\mathbf{x}}^i_2 > \mathring{\mathbf{x}}^i_2$ — *i.e.*, $\widetilde{\mathbf{x}}^i_1$ is not feasible for i.

Next we prove that

$$(*) \qquad \text{if } \widetilde{\mathbf{x}}^i \succsim^i \widehat{\mathbf{x}}^i, \text{ then } \widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i \geqq \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i :$$

Assume that $\widetilde{\mathbf{x}}^i \succsim^i \widehat{\mathbf{x}}^i$ and suppose that $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i < \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i$. Since $\succsim^i$ is LNS, there is a bundle $\overline{\mathbf{x}}$ that satisfies both $\widehat{\mathbf{p}} \cdot \overline{\mathbf{x}} < \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i$ and $\overline{\mathbf{x}} \succ^i \widetilde{\mathbf{x}}^i$, and therefore $\overline{\mathbf{x}} \succ^i \widehat{\mathbf{x}}^i$, which violates (U-Max). Therefore we must have $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i \geqq \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i$.

Now we can prove that the two-person coalition cannot improve upon the allocation $(\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)$ — *i.e.*, that $(\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)$ is Pareto optimal: Assume that $(\widetilde{\mathbf{x}}^1, \widetilde{\mathbf{x}}^2)$ is a Pareto improvement upon $(\widehat{\mathbf{x}}^1, \widehat{\mathbf{x}}^2)$, and wlog assume that $\widetilde{\mathbf{x}}^1 \succ^1 \widehat{\mathbf{x}}^1$ and $\widetilde{\mathbf{x}}^2 \succsim^2 \widehat{\mathbf{x}}^2$. Therefore (U-Max) yields $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^1 > \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^1$ and $(*)$ yields $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^2 \geqq \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^2$. Therefore $\widehat{\mathbf{p}} \cdot (\widetilde{\mathbf{x}}^1 + \widetilde{\mathbf{x}}^2) > \widehat{\mathbf{p}} \cdot (\mathring{\mathbf{x}}^1 + \mathring{\mathbf{x}}^2)$, and since $\widehat{\mathbf{p}} \in \mathbb{R}^\ell_{++}$, this implies that $\widetilde{\mathbf{x}}^1_k + \widetilde{\mathbf{x}}^2_k > \mathring{\mathbf{x}}^1_k + \mathring{\mathbf{x}}^2_k$ for some good k — *i.e.*, $(\widetilde{\mathbf{x}}^1, \widetilde{\mathbf{x}}^2)$ is not feasible for the two-person coalition. $\|$