ECON 501B FALL 2015
MIDTERM EXAM
SOLUTIONS

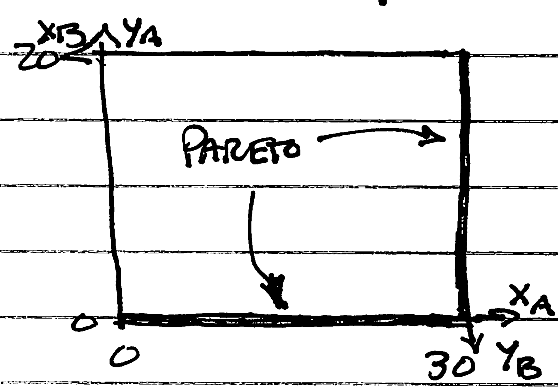
(1) $u_A(x,y) = 3x + y$ AND $u_B(x,y) = x + y$; $MRS_A = 3$, $MRS_B = 1$.

(a) WE HAVE $MRS_A > MRS_B$ EVERYWHERE, SO NO INTERIOR ALLOCATIONS ARE PARETO. BECAUSE $MRS_A > MRS_B$, A PARETO IMPROVEMENT IS MADE IF AND ONLY IF WE TRANSFER SOME OF THE X-GOOD FROM B TO A AND SOME OF THE Y-GOOD FROM A TO B (TRANSFERING JUST ONE GOOD REDUCES SOMEONE'S UTILITY). THIS IS POSSIBLE IF AND ONLY IF $x_B > 0$ AND $y_A > 0$, SO THOSE ALLOCATIONS AREN'T PARETO, AND IF EITHER $x_B = 0$ OR $y_A = 0$ THEN THE ALLOCATION IS PARETO.

ALTERNATIVELY, IN TERMS OF THE FOMC FOR THE P-MAX PROBLEM:

IF $x_B > 0$ AND $y_A > 0$, THEN WE MUST HAVE $u_x^B = \sigma_x$ AND $u_y^A = \sigma_y$ FOR SOME $\sigma_x, \sigma_y > 0$, AND ALSO $u_y^B \leqq \sigma_y$ AND $u_x^A \leqq \sigma_x$; $\therefore MRS_A \leqq \frac{\sigma_x}{\sigma_y} \leqq MRS_B$, WHICH IS VIOLATED EVERYWHERE ($MRS_A > MRS_B$).

IF $x_B = 0$ OR $y_A = 0$, THEN WE MUST HAVE $u_y^A \leqq \sigma_y$ AND $u_x^B \leqq \sigma_x$ FOR SOME $\sigma_x, \sigma_y > 0$, AND ALSO $u_x^A = \sigma_x$ AND $u_y^B = \sigma_y$ (BECAUSE $x_A > 0$ AND $y_B > 0$), i.e., $MRS_A \geqq \frac{\sigma_x}{\sigma_y} \geqq MRS_B$, WHICH IS SATISFIED IF $1 \leqq \frac{\sigma_x}{\sigma_y} \leqq 3$.



(b) THE FIRST WELFARE THEOREM REQUIRES ONLY THAT EACH PREFERENCE BE LNS. BOTH u_A AND u_B ARE LNS HERE, SO THE THEOREM ENSURES THAT ANY WALRASIAN EQUILIBRIUM ALLOCATION IS PARETO. THEREFORE NO NON-PARETO ALLOCATIONS CAN BE SUPPORTED AS EQUILIBRIA.

(c) IN (a) WE FOUND THAT EFFICIENCY PRICES σ_x, σ_y AT ANY PARETO ALLOCATION MUST SATISFY $1 \leqq \frac{\sigma_x}{\sigma_y} \leqq 3$, SO WE SHOULD EXPECT EQUILIBRIUM PRICES p_x, p_y TO SATISFY $1 \leqq \rho \leqq 3$, WHERE $\rho = \frac{p_x}{p_y}$. IF $1 < \rho < 3$, THEN AMY CHOOSES THE BUNDLE $(x_A, y_A) = (\mathring{x}_A, \mathring{y}_A) = (30, 0)$ AND BEV CHOOSES $(x_B, y_B) = (\mathring{x}_B, \mathring{y}_B) = (0, 20)$, SO THIS IS AN EQUILIBRIUM. IF $\rho = 1$ AMY STILL CHOOSES $(x_A, y_A) = (30, 0)$, AND BEV IS INDIFFERENT AMONG BUNDLES ON HER BUDGET CONSTRAINT $x_B + y_B = \mathring{x}_B + \mathring{y}_B = 20$, SO $(x_B, y_B) = (0, 20)$ STILL MAXIMIZES u_B ON HER BUDGET CONSTRAINT. SIMILARLY, IF $\rho = 3$ THEN BEV CHOOSES $(0, 20)$ AND AMY IS INDIFFERENT AMONG BUNDLES ON HER BUDGET CONSTRAINT, WHICH INCLUDE $(x_A, y_A) = (\mathring{x}_A, \mathring{y}_A) = (30, 0)$. SO THE INITIAL ALLOCATION, WITH ANY PRICES p_x, p_y THAT SATISFY $1 \leqq \frac{p_x}{p_y} \leqq 3$, CONSTITUTE A WALRASIAN EQUILIBRIUM.

TO SEE THAT THERE ARE NO OTHER EQUILIBRIA: IF $\rho < 1$, THEN BOTH CONSUMERS CHOOSE $y_i = 0$, WHICH IS NOT AN EQUILIBRIUM (EXCESS SUPPLY OF y-GOOD); IF $\rho > 3$, THEN BOTH CHOOSE $x_i = 0$, WHICH IS NOT AN EQUILIBRIUM.

(d) ASSUME THAT $(\mathring{x}_A, \mathring{y}_A) = (\mathring{x}_B, \mathring{y}_B) = (15, 10)$. AS IN (c), AN EQUILIBRIUM MUST HAVE $1 \leqq \rho \leqq 3$. BUT NOW IF $\rho > 1$, THEN B CHOOSES THE BUNDLE (x_B, y_B) THAT SATISFIES $x_B = 0$ AND $p_y y_B = p_x \mathring{x}_B + p_y \mathring{y}_B = 15p_x + 10p_y$; i.e., $y_B = 15\rho + 10 > 25$. BUT $\mathring{y} = 20$, SO THERE IS EXCESS DEMAND FOR THE y-GOOD — THIS IS NOT AN EQUILIBRIUM.

IF $\rho = 1$, THEN A CHOOSES $y_A = 0$ AND $x_A = \mathring{x}_A + \mathring{y}_A = 25$; THE MARKET CLEARS IF $(x_B, y_B) = (5, 20)$. BEV IS INDIFFERENT AMONG THE BUNDLES ON HER BUDGET CONSTRAINT $x_B + y_B = \mathring{x}_B + \mathring{y}_B = 25$, SO u_B IS INDEED MAXIMIZED (BUT NOT UNIQUELY) AT $(x_B, y_B) = (5, 20)$.

SO THE ONLY EQUILIBRIUM PRICES SATISFY $p_x = p_y$. THE ONLY EQUILIBRIUM ALLOCATION IS $(x_A, y_A) = (25, 0)$ AND $(x_B, y_B) = (5, 20)$.

(2) $u_A(x_A, y_A, x_B) = 3x_A + y_A + 10\log x_B$.

(a) $\max u_A(x_A, y_A, x_B)$ s.t. $x_A, y_A, x_B, y_B \geq 0$ AND TO * SEE BELOW

$$x_A + y_A \leq 30 \quad : \sigma_x$$
$$x_B + y_B \leq 20 \quad : \sigma_y$$
$$x_B + y_B \geq u_B \quad : \lambda$$

FOMC: (INTERIOR)

$$\left.\begin{array}{ll} x_A: & 3 = \sigma_x \\ y_A: & 1 = \sigma_y \\ x_B: & \frac{10}{x_B} = \sigma_x - \lambda \\ y_B: & 0 = \sigma_y - \lambda \end{array}\right\}$$

$\sigma_x = 3,\ \sigma_y = 1,\ \lambda = \sigma_y = 1$

$\therefore \frac{10}{x_B} = 2$, i.e., $x_B = 5$.

$\therefore$ THE INTERIOR PARETO ALLOCATIONS SATISFY $x_A = 25, x_B = 5$ AND $y_A + y_B = 20$.

(b) THE WALRASIAN EQUILIBRIA ARE THE SAME AS IN #1(c): NO TRADE — i.e., $(x_i, y_i) = (\mathring{x}_i, \mathring{y}_i)$ FOR $i = A, B$, AND PRICES THAT SATISFY $1 \leq \frac{p_x}{p_y} \leq 3$.

* ALTERNATIVE APPROACH FOR (a):

$u_A = 3x_A + y_A + 10\log(30 - x_A)$

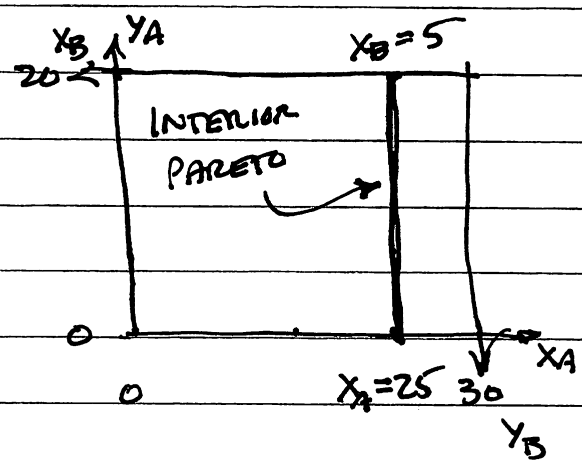
so $MRS_A = 3 - \frac{10}{30 - x_A}$

$MRS_A = MRS_B$:

$$3 - \frac{10}{30 - x_A} = 1$$

i.e., $\frac{10}{30 - x_A} = 2$; $30 - x_A = \frac{10}{2} = 5$

$x_A = 25, x_B = 5$.



(3) $u_A(x,y) = \sqrt{x} + \sqrt{y}$ AND $u_B(x,y) = \min\{x,y\}$; $\mathring{x} = \mathring{y}$.

(a) PARETO EFFICIENCY REQUIRES THAT $x_B = y_B$; OTHERWISE WE COULD TRANSFER SOME OF THE GOOD B HAS MORE OF TO A, MAKING A BETTER OFF AND B NO WORSE OFF. SO THE PARETO ALLOCATIONS ARE THE ONES THAT SATISFY

$x_A = y_A$, $x_B = y_B$, AND $x_A + x_B = y_A + y_B = \mathring{x} = \mathring{y}$.

(b) SINCE PARETO REQUIRES $x_A = y_A$ AND $x_B = y_B$, WE HAVE $u_A = 2\sqrt{x_A}$ AND $u_B = x_B$, AND WE ALSO HAVE $x_A + x_B = \mathring{x} = 16$. FROM $u_A = 2\sqrt{x_A}$ WE OBTAIN $u_A^2 = 4x_A$, i.e., $x_A = \frac{1}{4}u_A^2$. SO WE HAVE

$\frac{1}{4}u_A^2 + u_B = 16$ FOR THE UTILITY FRONTIER.

(c) WE HAVE $u_B = 16 - \frac{1}{4}u_A^2$ AND WE WANT TO

$\max W = u_A + u_B = u_A + 16 - \frac{1}{4}u_A^2$. ← NOTE THAT THIS IS STRICTLY CONCAVE IN u_A.

FOMC: $1 - \frac{1}{2}u_A = 0$; i.e., $u_A = 2$; $\therefore u_B = 16 - \frac{1}{4}(4) = 15$.

$\therefore 2\sqrt{x_A} = 2$, i.e., $x_A = 1$ AND $x_B = 15$:

$(x_A, y_A) = (1,1)$ AND $(x_B, y_B) = (15,15)$.

EQUIVALENTLY, $\max W(u_A, u_B) = u_A + u_B$ SUBJECT TO

$\frac{1}{4}u_A^2 + u_B = 16$. ← STRICTLY QUASICONVEX CONSTRAINT

FOMC:

$$\left.\begin{array}{l} u_A: 1 = \frac{1}{2}u_A\lambda \\ u_B: 1 = \lambda \end{array}\right\} \quad \frac{1}{2}u_A\lambda = \lambda;\ \text{i.e., } \frac{1}{2}u_A = 1;\ \text{i.e., } u_A = 2,$$

THEN SAME AS ABOVE.

(d) THE FIRST WELFARE THEOREM ENSURES THAT AN EQUILIBRIUM ALLOCATION MUST SATISFY $x_A = y_A$ AND $x_B = y_B$ (FROM (a)), AND THEREFORE $MRS_A = 1$, AND THEREFORE $p_x = p_y$. THE BUDGET CONSTRAINTS THEREFORE YIELD

$$x_A + y_A = \mathring{x}_A + \mathring{y}_A = 13, \text{ so } (x_A, y_A) = (6\tfrac{1}{2}, 6\tfrac{1}{2}),$$

$$x_B + y_B = \mathring{x}_B + \mathring{y}_B = 19, \text{ so } (x_B, y_B) = (9\tfrac{1}{2}, 9\tfrac{1}{2}).$$

(e) WE MUST HAVE $u_A \geqq \mathring{u}_A = 3+2 = 5$ AND $u_B \geqq \mathring{u}_B = 7$:

$$u_A = 2\sqrt{x_A} \geqq 5, \text{ i.e., } 4x_A \geqq 25, \text{ i.e., } x_A \geqq \frac{25}{4} = 6\tfrac{1}{4},$$

$$u_B = x_B = 7.$$

SO THE CORE ALLOCATIONS ARE THE ONES THAT SATISFY $\underset{\geqq \frac{25}{4}}{x_A = y_A}$, $\underset{\geqq 7}{x_B = y_B}$.