ECON 501B 2015 FINAL EXAM
SOLUTIONS

(1)(a) $\max_{x, y_A, y_B, y_C} u^A(x, y_A)$ s.t. $x, y_A, y_B, y_C \geqq 0$ AND

$$y_A + y_B + y_C \leqq \mathring{y} \quad : \sigma$$
$$u^B(x, y_B) \geqq u_B \quad : \lambda_B$$
$$u^C(x, y_C) \geqq u_C \quad : \lambda_C$$

FOMC (INTERIOR): $\exists\, \sigma, \lambda_B, \lambda_C \geqq 0$ s.t.

x: $\quad u^A_x = -\lambda_B u^B_x - \lambda_C u^C_x$; i.e., $u^A_x + \lambda_B u^B_x + \lambda_C u^C_x = 0$

y_A: $\quad u^A_y = \sigma$

y_B: $\quad 0 = \sigma - \lambda_B u^B_y$; i.e., $\lambda_B = \frac{\sigma}{u^B_y}$

y_C: $\quad 0 = \sigma - \lambda_C u^C_y$; i.e., $\lambda_C = \frac{\sigma}{u^C_y}$

COMBINING: $\quad u^A_x + \sigma \frac{u^B_x}{u^B_y} + \sigma \frac{u^C_x}{u^C_y} = 0$

i.e., $\frac{u^A_x}{u^A_y} + \frac{u^B_x}{u^B_y} + \frac{u^C_x}{u^C_y} = 0$; i.e., $\boxed{MRS^A + MRS^B + MRS^C = 0.}$

(b) A'S DECISION PROBLEM:

$$\max_{m_A} u^A\left(\tfrac{1}{3}(m_A + m_B + m_C),\ \mathring{y}_A - (m_B - m_C)\tfrac{1}{3}(m_A + m_B + m_C)\right)$$

FOMC: $\frac{1}{3} u^A_x - \frac{1}{3}(m_B - m_C) u^A_y = 0 \quad (\leqq 0 \text{ IF } m_A = 0).$

i.e., $u^A_x = (m_B - m_C) u^A_y$; i.e., $\frac{u^A_x}{u^A_y} = m_B - m_C$

i.e., $MRS^A = m_B - m_C \quad (\leqq \text{ IF } m_A = 0)$

SAME FOR B AND C: $MRS^B = m_C - m_A$, $MRS^C = m_A - m_B$.

$\therefore$ IF EACH MAXIMIZES u^i TAKING OTHERS' m_i'S AS GIVEN: $MRS^A + MRS^B + MRS^C = m_B - m_C + m_C - m_A + m_A - m_B$
$= 0.$

FOR (c)–(f): $MRS^A = 4 - x$, $MRS^B = 1 - x$, $MRS^C = 1 - x$.

(c) $y_A + y_B + y_C = \mathring{y} = 300$ AND $\sum MRS^i = 0$ – i.e.,
$4 - x + 1 - x + 1 - x = 6 - 3x = 0$, SO $x = 2$; WRITE $\hat{x} = 2$.

(d) $\forall i: \beta_i = a_i$, SO $\beta_A = 4$ AND $\beta_B = \beta_C = 1$.
IF $m_i = \beta_i$ $(\forall i)$, THEN $\bar{m} = \bar{\beta} = \bar{a} = 2$, SO $x = 2 = \hat{x}$
WHICH IS PARETO EFFICIENT (IF $y_A + y_B + y_C = \mathring{y}$).
IS A MAXIMIZING? WE HAVE $m_B - m_C = \beta_B - \beta_C = 0$,
BUT AT $x = 2$ WE HAVE $MRS^A = 2$, SO A IS NOT MAXIMIZING.
SIMILARLY, $m_C - m_A = 1 - 4 = -3$ AND $MRS^B = -1$,
AND $m_A - m_B = 4 - 1 = 3$ AND $MRS^C = -1$.
SO NOONE IS MAXIMIZING, TAKING OTHERS' m_i'S AS GIVEN.
$\therefore$ THIS IS NOT A NASH EQUILIBRIUM.

(e) SUPPOSE $x = 1$; THEN $MRS^A = 3$ AND $MRS^B = MRS^C = 0$. THIS VIOLATES THE CONDITION $\sum MRS^i = 0$, SO IT CANNOT BE AN INTERIOR PARETO ALLOCATION. BUT WHAT IF SOME $y_i = 0$? WE HAVE $MRS^A = 3$, SO A WOULD PAY TO INCREASE x. BUT SUPPOSE $y_A = 0$: THEN WE CAN'T TRANSFER ANY DOLLARS FROM A TO B AND C TO COMPENSATE THEM FOR THE DECREASE IN u^B AND u^C CAUSED BY THE INCREASE IN x. SO ANY ALLOCATION THAT SATISFIES $x = 1$ (BEST x FOR B AND C), $y_A = 0$, AND $y_B + y_C = \mathring{y} = 300$ IS PARETO.

SIMILARLY, IF $x = 4$ (BEST x FOR A) AND $y_B = y_C = 0$ AND $y_A = \mathring{y} = 300$, THAT'S ALSO PARETO EFFICIENT.

(f) NE THAT YIELD INTERIOR ALLOCATIONS MUST HAVE $x = \hat{x} = 2$, SO WE HAVE $MRS^A = 2$, $MRS^B = MRS^C = -1$.

THEREFORE
$$m_B - m_C = 2$$
$$m_C - m_A = -1$$
$$m_A - m_B = -1.$$
AND $m_A + m_B + m_C = 6$.

THEREFORE $m_B = m_A + 1$ AND $m_C = m_A - 1$, SO
$$m_A + m_B + m_C = m_A + m_A + 1 + m_A - 1 = 3m_A,$$
AND THEREFORE $3m_A = 6$, i.e., $m_A = 2$, SO $m_B = 3$, $m_C = 1$.

WE HAVE $\boxed{(m_A, m_B, m_C) = (2, 3, 1)}$.

CHECKING:
$$m_B - m_C = 2 = MRS^A$$
$$m_C - m_A = -1 = MRS^B$$
$$m_A - m_B = -1 = MRS^C.$$

THIS IS THE ONLY SOLUTION OF THE FOUR EQUATIONS ABOVE, SO IT IS THE ONLY INTERIOR NE.

(2) $u(x,y)=\sqrt{x}+\sqrt{y}$, $\quad MRS = \dfrac{\sqrt{y}}{\sqrt{x}}$.

(a) INTERIOR PARETO REQUIRES $MRS^1 = \cdots = MRS^n = r$, SAY;

i.e., $\dfrac{\sqrt{y_i}}{\sqrt{x_i}} = r$; $\dfrac{y_i}{x_i} = r^2$; $y_i = r^2 x_i$ FOR ALL i

$\therefore \sum y_i = r^2 \sum x_i$, i.e., $\mathring{y} = r^2 \mathring{x}$ FOR PARETO, i.e., $r^2 = \dfrac{\mathring{y}}{\mathring{x}}$.

FOR EACH i: $u_i = \sqrt{x_i} + \sqrt{y_i} = \sqrt{x_i} + \sqrt{r^2 x_i} = (1+r)\sqrt{x_i}$

$$\therefore u_i^2 = (1+r)^2 x_i, \ \forall i$$

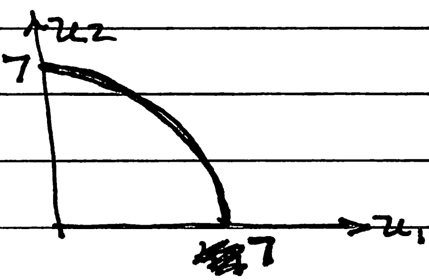
$$\sum u_i^2 = (1+r)^2 \sum x_i = (1+r)^2 \mathring{x}.$$

$$(1+r)^2 = 1 + 2r + r^2 = 1 + 2\frac{\sqrt{\mathring{y}}}{\sqrt{\mathring{x}}} + \frac{\mathring{y}}{\mathring{x}},$$

SO $\sum u_i^2 = \mathring{x} + 2\sqrt{\mathring{x}}\sqrt{\mathring{y}} + \mathring{y} = \left(\sqrt{\mathring{x}} + \sqrt{\mathring{y}}\right)^2$.

IF $n=2$ AND $(\mathring{x}, \mathring{y}) = (16, 9)$:

$$u_1^2 + u_2^2 = (4+3)^2 = 49$$



(c) $V(\cdot)$ IS THE VALUE FUNCTION FOR THE P-MAX PROBLEM

$$\max_{x_1,y_1,x_2,y_2} u^1(x_1,y_1) \text{ s.t. } x_1, y_1, x_2, y_2 \geqq 0$$
$$x_1 + x_2 \leqq \mathring{x}, \quad y_1 + y_2 \leqq \mathring{y}, \quad u^2(x_2,y_2) \geqq u_2.$$

THE SOLUTION FUNCTION GIVES $((x_1,y_1),(x_2,y_2))$, THE SOLUTION OF (P-MAX), AS A FUNCTION OF u_2.

(b) ON THE FOLLOWING PAGE.

(b) LET $\mathcal{F}$ DENOTE THE SET OF FEASIBLE ALLOCATIONS AND $\mathcal{U}$ THE SET OF FEASIBLE UTILITY PROFILES (u_1, u_2):

$$\mathcal{F} = \{((x_1,y_1),(x_2,y_2)) \in \mathbb{R}^4_+ \mid (x_1,y_1)+(x_2,y_2) \leqq (\mathring{x}, \mathring{y})\},$$

$$\mathcal{U} = \{(u_1,u_2) \in \mathbb{R}^2 \mid \exists ((x_1,y_1),(x_2,y_2)) \in \mathcal{F} \text{ s.t. } u_i = u(x_i,y_i), i=1,2\}.$$

NOTE THAT $\mathcal{U} = \{(u_1,u_2) \in \mathbb{R}^2 \mid u_1^2 + u_2^2 \leqq c := (\sqrt{\mathring{x}} + \sqrt{\mathring{y}})^2\}$.

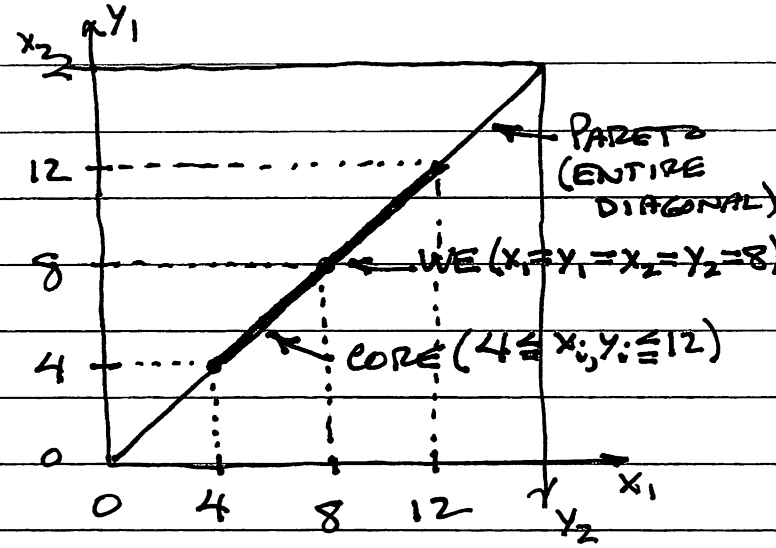
SUPPOSE THAT $(\hat{u}_1, \hat{u}_2)$ MAXIMIZES $\alpha_1 u_1 + \alpha_2 u_2$ FOR SOME $\alpha_1, \alpha_2 > 0$. IF $(\hat{u}_1, \hat{u}_2)$ IS NOT PARETO, THEN $\exists (u_1,u_2) \in \mathcal{U}$ s.t. $u_1 \geqq \hat{u}_1$, $u_2 \geqq \hat{u}_2$ AND $u_1 > \hat{u}_1$ OR $u_2 > \hat{u}_2$. THEN $\alpha_1 u_1 + \alpha_2 u_2 > \alpha_1 \hat{u}_1 + \alpha_2 \hat{u}_2$, CONTRADICTING OUR ASSUMPTION THAT $(\hat{u}_1, \hat{u}_2)$ MAXIMIZES $\alpha_1 u_1 + \alpha_2 u_2$ ON $\mathcal{U}$. NOW SUPPOSE (WLOG) THAT $\alpha_1 > 0$ AND $\alpha_2 = 0$; THEN MAXIMIZING $\alpha_1 u_1 + \alpha_2 u_2$ REQUIRES MAXIMIZING u_1, WHICH REQUIRES THAT $(x_1,y_1) = (\mathring{x}, \mathring{y})$, WHICH IS PARETO EFFICIENT.

SUPPOSE THAT $((\hat{x}_1,\hat{y}_1),(\hat{x}_2,\hat{y}_2))$ IS PARETO AND LET $\hat{u}_1 = u(\hat{x}_1,\hat{y}_1)$ AND $\hat{u}_2 = u(\hat{x}_2,\hat{y}_2)$. THEN $\hat{u}_1^2 + \hat{u}_2^2 = (\sqrt{\mathring{x}} + \sqrt{\mathring{y}})^2$ — $(\hat{u}_1, \hat{u}_2)$ IS ON THE BOUNDARY OF $\mathcal{U}$. SINCE THIS PARTICULAR $\mathcal{U}$ IS CONVEX, THERE IS A SUPPORTING HYPERPLANE $H = \{(u_1,u_2) \mid \alpha_1 u_1 + \alpha_2 u_2 = b\}$ FOR SOME $b \in \mathbb{R}$ AND $\alpha_1, \alpha_2 \geqq 0$ THAT INCLUDES THE BOUNDARY POINT $(\hat{u}_1, \hat{u}_2)$. SINCE H SUPPORTS $\mathcal{U}$ WE HAVE $(u_1,u_2) \in \mathcal{U} \Rightarrow \alpha_1 u_1 + \alpha_2 u_2 \leqq \alpha_1 \hat{u}_1 + \alpha_2 \hat{u}_2$ — i.e., $(\hat{u}_1, \hat{u}_2)$ MAXIMIZES $\alpha_1 u_1 + \alpha_2 u_2$ ON $\mathcal{U}$.

(d) WE: $\begin{cases} p_x = p_y \\ x_1 = y_1 = x_2 = y_2 = 8 \end{cases}$

PARETO: $\begin{cases} x_1 = y_1 \text{ \& } x_2 = y_2 \\ x_1 + x_2 = y_1 + y_2 = 16 \end{cases}$

CORE: $x_1 + x_2 = y_1 + y_2 = 16$,
$x_1 = y_1$ & $x_2 = y_2$,
AND $u_i \geqq \mathring{u}_i = 4$ $(i = 1,2)$
i.e., $2\sqrt{x_i} \geqq 4$;
i.e., $\sqrt{x_i} \geqq 2$; $x_i \geqq 4$
$\therefore 4 \leqq x_i, y_i \leqq 12$ $(i = 1,2)$



(e) WE: $p_x = p_y$, $x_i = y_i = 8$ $(i = 1,2)$, $(x_3, y_3) = (9,9)$.

(f) CORE ALLOCATIONS ALL HAVE $MRS^i = 1$ – i.e., $x_i = y_i$
– FOR EACH i, BECAUSE THEY'RE PARETO EFFICIENT.
SUPPOSE $(x_3, y_3) > (9,9)$; THEN $x_1 + x_2 < 16$ AND $y_1 + y_2 < 16$;
THE COALITION $S = \{1,2\}$ CAN IMPROVE UPON THIS
BY SIMPLY ALLOCATING $(\mathring{x}, \mathring{y})_S = (16, 16)$ TO INCREASE
u_1 AND u_2. SO NO CORE ALLOCATION HAS $x_3, y_3 > 9$.
SUPPOSE $(x_3, y_3) < (9,9)$; THEN THE ONE-PERSON
COALITION CAN IMPROVE BY SIMPLY CHOOSING
$(x_3, y_3) = (\mathring{x}_3, \mathring{y}_3) = (9,9)$. SO NO CORE ALLOCATION
HAS $x_3, y_3 < 9$. <u>EVERY</u> CORE ALLOCATION SATISFIES
$(x_3, y_3) = (9,9)$.

(g) FOR THE COALITION $S=\{1,3\}$ WE HAVE

$$(\mathring{x},\mathring{y})_S = (\mathring{x}_1,\mathring{y}_1)+(\mathring{x}_3,\mathring{y}_3) = (16,0)+(9,9) = (25,9).$$

THEREFORE THE UTILITY FRONTIER FOR S IS

$$u_1^2+u_3^2 = (\sqrt{25}+\sqrt{9})^2 = (5+3)^2 = 64.$$

THEREFORE ANY CORE ALLOCATION MUST SATISFY $u_1^2+u_3^2 \geqq 64$. SINCE $(x_3,y_3)=(9,9)$, WE HAVE $u_3=\sqrt{9}+\sqrt{9}=3+3=6$ AND $u_3^2=36$; THEREFORE ANY CORE ALLOCATION IN WHICH $(x_3,y_3)=(9,9)$ MUST SATISFY $u_1^2 \geqq 64-36=28$, i.e., $u_1 \geqq \sqrt{28}=2\sqrt{7}$. SINCE A CORE ALLOCATION ALSO HAS TO SATISFY $x_1=y_1$, WE HAVE $u_1=2\sqrt{x_1}=2\sqrt{y_1}$; THEREFORE $x_1,y_1 \geqq 7$. SINCE $(\mathring{x},\mathring{y})=(25,25)$ AND $x_3=9$, $x_1\geqq 7$, WE HAVE $x_2 \leqq 9$ AND $y_2 \leqq 9$.

THE SAME ARGUMENT APPLIED TO $S=\{2,3\}$ YIELDS $x_2,y_2 \geqq 7$ AND $x_1,y_1 \leqq 9$ AS WELL.

THEREFORE THE CORE ALLOCATIONS ARE THE ONES THAT SATISFY

~~$x_3=y_3=9$ AND (JUST AS IN (d)) $x_1+x_2=y_1+y_2=16$, AND (UNLIKE IN (d)) $7\leqq x_1 \leqq 9$ AND $7 \leqq x_2 \leqq 9$.~~

$(x_3,y_3)=(9,9)$,

$(x_1,y_1)+(x_2,y_2)=(16,16)$, JUST AS IN (d)

$7 \leqq x_1 \leqq 9$ AND $7 \leqq x_2 \leqq 9$, UNLIKE IN (d).

(3) $u^A(x_0, x_D, x_R) = x_0 + 3\sqrt{x_D} + 2\sqrt{x_R}$ $\quad (\mathring{x}_0^A, \mathring{x}_D^A, \mathring{x}_R^A) = (12, 15, 15)$

$u^B(x_0, x_D, x_R) = x_0 + 4\sqrt{x_D} + \sqrt{x_R}$ $\quad (\mathring{x}_0^B, \mathring{x}_D^B, \mathring{x}_R^B) = \underline{(12, 10, 30)}$

$(\mathring{x}_0, \mathring{x}_D, \mathring{x}_R) = (24, 25, 45).$

$$MRS_D^A = \frac{3}{2\sqrt{x_D^A}}, \quad MRS_R^A = \frac{1}{\sqrt{x_R^A}}$$

$$MRS_D^B = \frac{2}{\sqrt{x_D^B}}, \quad MRS_R^B = \frac{1}{2\sqrt{x_R^B}}$$

(a) $MRS_D^A = MRS_D^B : \frac{3}{2\sqrt{x_D^A}} = \frac{2}{\sqrt{x_D^B}}$; i.e., $3\sqrt{x_D^B} = 4\sqrt{x_D^A}$; $9x_D^B = 16x_D^A$.

$MRS_R^A = MRS_R^B : \frac{1}{\sqrt{x_R^A}} = \frac{1}{2\sqrt{x_R^B}}$; i.e., $2\sqrt{x_R^B} = \sqrt{x_R^A}$; $4x_R^B = x_R^A$.

$\therefore$ $x_D^A = 9$ $\quad x_R^A = 36$ $\quad MRS_D^A = \frac{1}{2} = MRS_D^B$

$\underline{x_D^B = 16}$ $\quad \underline{x_R^B = 9}$ $\quad MRS_R^A = \frac{1}{6} = MRS_R^B$

$x_D^A + x_D^B = 25 = \mathring{x}_D$, $x_R^A + x_R^B = 45 = \mathring{x}_R$ $\quad x_0^A + x_0^B = 24$

(b) $P_D = \frac{1}{2}$, $P_R = \frac{1}{6}$; $x_D^A, x_R^A, x_D^B, x_R^B$ AS IN (a).

$x_0^A = 12 - \frac{1}{2}(9-15) - \frac{1}{6}(36-15) = 12 + 3 - \frac{21}{6} = 15 - 3\frac{1}{2} = 11\frac{1}{2}.$

$x_0^B = 12 - \frac{1}{2}(16-10) - \frac{1}{6}(9-30) = 12 - 3 + \frac{21}{6} = 9 + 3\frac{1}{2} = \underline{12\frac{1}{2}}$

$x_0^A + x_0^B = 24 = \mathring{x}_0$

(c) $\frac{1}{1+r} = MRS_D^i + MRS_R^i = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$; $\therefore 1+r = \frac{3}{2}$, $r = \frac{1}{2} = 50\%$

(d) $d_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $d_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$q_1 = P_D + P_R = \frac{2}{3}$, $q_2 = 3P_D + 2P_R = \frac{3}{2} + \frac{2}{6} = \frac{11}{6}$.

... NEXT PAGE ...

(d) $d_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$q_1 = p_D + p_R = \frac{2}{3}, \quad q_2 = 3p_D + 2p_R = \frac{3}{2} + \frac{2}{6} = \frac{9}{6} + \frac{2}{6} = \frac{11}{6}$$

$$\text{For } A: \begin{bmatrix} x_D^A - \overset{o}{x}{}_D^A \\ x_R^A - \overset{o}{x}{}_R^A \end{bmatrix} = \begin{bmatrix} -6 \\ 21 \end{bmatrix} = y_1^A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2^A \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$\therefore y_1^A = 75, \; y_2^A = -27$

$$q_1 y_1^A + q_2 y_2^A = \frac{2}{3}(75) + \frac{11}{6}(-27)$$
$$= 50 - 11(4\tfrac{1}{2}) = 50 - 49\tfrac{1}{2} = \tfrac{1}{2}$$

$y_1^B = -75, \; y_2^B = 27$

$$q_1 y_1^B + q_2 y_2^B = \frac{2}{3}(-75) + \frac{11}{6}(27)$$
$$= -50 + 49\tfrac{1}{2} = -\tfrac{1}{2}$$

SPENDING PLANS (x_0^A, x_D^A, x_R^A) AND (x_0^B, x_D^B, x_R^B) THE SAME AS IN (b), INTEREST RATE THE SAME AS IN (b): $\frac{1}{1+r} = q_1$.

(e) u^A AND u^B CAN BOTH BE REPRESENTED AS VN-M EXPECTED UTILITY:

$$u^A(x_0, x_D, x_R) = x_0 + \frac{3}{5} v^A(x_D) + \frac{2}{5} v^A(x_R)$$
$$u^B(x_0, x_D, x_R) = x_0 + \frac{4}{5} v^B(x_D) + \frac{1}{5} v^B(x_R),$$

WHERE $v^A(z) = v^B(z) = 5\sqrt{z} = \sqrt{25z}$.