ECON 501B MIDTERM EXAM SOLUTIONS
FALL 2014

(1) $u(x,y)=xy$ FOR BOTH A AND B. $\mathring{x}_A=\mathring{x}_B=8$ AND $\mathring{y}=0$.

$q_1=f_1(z_1)=2\sqrt{z_1}$, $f_1'(z_1)=\frac{1}{\sqrt{z_1}}$.

$q_2=f_2(z_2)=\frac{1}{2}z_2$, $f_2'(z_2)=\frac{1}{2}$.

WE HAVE $f_1'(z_1)=f_2'(z_2)=\frac{1}{2}$ WHEN $z_1=4, q_1=4$.

WHEN $q_1<4$, WE HAVE $z_1<4$, $f_1'(z_1)>f_2'(z_2)$ FOR ALL z_2, SO WE SHOULD USE JUST f_1.

WHEN $q_1>4$, WE HAVE $z_1>4$ AND $f_1'(z_1)<f_2'(z_2)$ FOR ALL z_2, SO WE SHOULD USE f_2 FOR ALL UNITS BEYOND $q=4$.

SO EFFICIENT PRODUCTION FOR THE ECONOMY REQUIRES:

TO GET $q\leqq 4$: $z_2=0$, $z_1=\frac{1}{4}q^2$.

TO GET $q>4$: $z_1=4, q_1=4, z_2=2(q-q_1)=2(q-4)=2q-8$.

<u>PARETO:</u>

IF WE PRODUCE $q<4$, THEN WE HAVE $z_2=q_2=0$, AND $z=z_1=\frac{1}{4}q^2\leqq 4$.

WE HAVE $f_1'(z_1)=\frac{1}{\sqrt{z_1}}\geqq\frac{1}{\sqrt{4}}=\frac{1}{2}$.

$\therefore$ PARETO EFFICIENCY REQUIRES $MRS_A=MRS_B=f_1'(z_1)\geqq\frac{1}{2}$.

THIS REQUIRES $y_A\geqq\frac{1}{2}x_A$ AND $y_B\geqq\frac{1}{2}x_B$, SO $y\geqq\frac{1}{2}x$.

WE HAVE $y=q\leqq 4$ AND

$$x=\mathring{x}-z=16-\frac{1}{4}q^2\geqq 16-\frac{1}{4}(16)=12.$$

$\therefore$ $y\leqq\frac{1}{3}x$, WHICH IS INCONSISTENT WITH $y\geqq\frac{1}{2}x$ FROM ABOVE.

$\therefore$ NO PARETO ALLOCATION HAS $q\leqq 4$ AND $z\leqq 4$.

SO WE HAVE $q > 4$ AND $z > 4$, AND $\therefore$ WE MUST HAVE $MRS_A = MRS_B = f_2'(z_2) = \frac{1}{2}$. THAT REQUIRES $y_A = \frac{1}{2} x_A$ AND $y_B = \frac{1}{2} x_B$.

$$\therefore\ y = q = \tfrac{1}{2}(x_A + x_B) = \tfrac{1}{2} x. \quad [\therefore\ x = 2q]$$

WE ALSO HAVE $z = z_1 + z_2 = 4 + (2q-8) = 2q - 4$.

WE HAVE $x + z = \mathring{x} = 16$; i.e., $2q + 4 + 2q - 8 = 16$;

i.e., $4q = 20$; i.e., $q = 5$.

$\therefore\ q = 5$; $z_1 = 4$, $z_2 = 2$; $q_1 = 4$, $q_2 = 1$;

$x_A + x_B = 16 - 6 = 10$ AND $y_A + y_B = q = 5$.

$y_A = \frac{1}{2} x_A$ AND $y_B = \frac{1}{2} x_B$.

~~(2) WALRASIAN EQUILIBRIUM~~

(2) WALRASIAN EQUILIBRIUM:

$p_x = 1$ (SAY), AND $p_y = 2$. $(z_1, q_1) = (4,4)$, $(z_2, q_2) = (2,1)$.

$\pi_1 = (2)(4) - (1)(4) = 8 - 4 = 4$ A OWNS THIS FIRM

$\pi_2 = (2)(1) - (1)(2) = 2 - 2 = 0$

A'S BUDGET CONSTRAINT: $x_A + 2y_A = \mathring{x}_A + \pi_1 = 8 + 4 = 12$

A CHOOSES (x_A, y_A) WITH $MRS_A = \frac{1}{2}$, SO $y_A = \frac{1}{2}x_A$;

$\therefore (x_A, y_A) = (6,3)$.

B'S BUDGET CONSTRAINT: $x_B + 2y_B = \mathring{x}_B = 8$.

B CHOOSES S.T. $y_B = \frac{1}{2}x_B$, SO WE HAVE

$(x_B, y_B) = (4,2)$.

WE HAVE $x_A + x_B = 6 + 4 = 10 = \mathring{x} - z = 16 - 6$,

$y_A + y_B = 3 + 2 = 5 = q = q_1 + q_2 = 4 + 1 = 5$,

SO BOTH MARKETS CLEAR.

(3) $u^A = \sqrt{xy}$, $MRS_A = \frac{y}{x}$ $\quad u^B = \min\{x, y\}$ $\quad \mathring{x} = \mathring{y}$

<u>PARETO:</u>

PARETO EFFICIENCY CLEARLY REQUIRES THAT $x_A = y_A$, $x_B = y_B$, AND $x_A + x_B = \mathring{x}$, $y_A + y_B = \mathring{y}$.

TO SEE THIS, NOTE THAT IF $x_B \neq y_B$ THEN THE LARGER COMPONENT CAN BE REDUCED WITHOUT REDUCING u_B, AND THE REDUCTION CAN BE USED TO AUGMENT x_A OR y_A, INCREASING u_A, SO THIS IS A PARETO IMPROVEMENT. IF $x_A \neq y_A$, THEN ALSO $x_B \neq y_B$, SO WE'RE AGAIN IN THE CASE ABOVE, FROM WHICH THERE'S A PARETO IMPROVEMENT.

TO SEE THAT ALL SUCH ALLOCATIONS <u>ARE</u> PARETO, NOTE THAT THE ONLY WAY TO INCREASE u_B IS TO INCREASE BOTH x_B AND y_B, WHICH ENTAILS A DECREASE IN BOTH x_A AND y_A, AND THEREFORE A DECREASE IN u_A. TO INCREASE u_A WE HAVE TO INCREASE EITHER x_A OR y_A, WHICH DECREASES EITHER x_B OR y_B, THEREBY DECREASING u_B.

(4) THE UTILITY FRONTIER:

WE MUST HAVE $x_A = y_A$ AND $x_B = y_B$, AS ABOVE.

$\therefore u_A = \sqrt{x_A^2} = x_A$ AND $u_B = \min\{x_B, y_B\} = x_B$.

$\therefore u_A + u_B = x_A + x_B = \mathring{x} = \mathring{y}$. SO THE UTILITY FRONTIER IS $u_A + u_B = \mathring{x} = \mathring{y}$.

(5) WALRASIAN EQUILIBRIUM: $(\mathring{x}_A, \mathring{y}_A) = (\mathring{x}, 0), (\mathring{x}_B, \mathring{y}_B) = (0, \mathring{y})$.

$p_x = p_y$; $x_A = y_A = x_B = y_B = \frac{1}{2}\mathring{x} = \frac{1}{2}\mathring{y}$.

VERIFY IT:

(M-CLR): $x_A + x_B = \frac{1}{2}\mathring{x} + \frac{1}{2}\mathring{x} = \mathring{x}$

$y_A + y_B = \frac{1}{2}\mathring{y} + \frac{1}{2}\mathring{y} = \mathring{y}$.

LET $p_x = p_y = 1$.

(U-MAX) FOR A:

BUDGET CONSTRAINT: $x_A + y_A = \mathring{x}_A$.

IF $x_A = y_A = \frac{1}{2}\mathring{x} = \frac{1}{2}\mathring{y}$, THEN $MRS_A = 1 = \frac{p_x}{p_y}$

AND $x_A + y_A = \frac{1}{2}\mathring{x} + \frac{1}{2}\mathring{y} = \frac{1}{2}\mathring{x} + \frac{1}{2}\mathring{x} = \mathring{x}$.

SO A IS MAXIMIZING $u^A(\cdot)$.

(U-MAX) FOR B:

BUDGET CONSTRAINT: $x_B + y_B = \mathring{y}_B$.

IF $x_B = y_B = \frac{1}{2}\mathring{x} = \frac{1}{2}\mathring{y}$, THEN $u_B = \frac{1}{2}\mathring{x} = \frac{1}{2}\mathring{y}$,

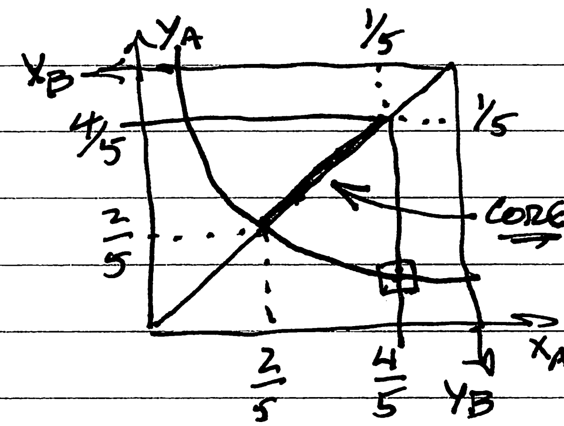
AND ~~IS~~ $u^B(\cdot)$ IS MAXIMIZED SUBJECT TO THE B.C., AS FOLLOWS: IF (WLOG) $x_B > y_B$ AND $x_B + y_B = \mathring{y}$, THEN $u_B = y_B$; BUT u_B IS LARGER AT $(\tilde{x}_B, \tilde{y}_B)$, WHERE $\tilde{x}_B = \tilde{y}_B = \frac{1}{2}(x_B + y_B)$.

⑥ $(\mathring{x}_A, \mathring{y}_A) = (\frac{4}{5}, \frac{1}{5})$ AND $(\mathring{x}_B, \mathring{y}_B) = (\frac{1}{5}, \frac{4}{5})$.

$\therefore \mathring{u}_A = \sqrt{(\frac{4}{5})(\frac{1}{5})} = \sqrt{\frac{4}{25}} = \frac{2}{5}$ AND $\mathring{u}_B = \min\{\frac{1}{5}, \frac{4}{5}\} = \frac{1}{5}$.

SO CORE ALLOCATIONS (WHICH MUST BE PARETO, SO $x_A = y_A$ AND $x_B = y_B$) ARE THE ONES THAT SATISFY $u_A \geq \mathring{u}_A = \frac{2}{5}$ AND $u_B \geq \mathring{u}_B = \frac{1}{5}$

— i.e., $x_A = y_A \geq \frac{2}{5}$ AND $x_B = y_B \geq \frac{1}{5}$



(7) $u^C(x,y) = \frac{1}{2}(x+y)$, $(\mathring{x}_C, \mathring{y}_C) = (1,1)$.

$S=\{A,B,C\}$: PARETO FOR S REQUIRES $x_i = y_i$ $(\forall i)$.

$\therefore u_A = x_A$, $u_B = x_B$, $u_C = x_C$.

PARETO ALSO REQUIRES $x_A + x_B + x_C = \mathring{x} = 2$,

SO WE HAVE $u_A + u_B + u_C = 2$ FOR THE UTILITY FRONTIER.

$S=\{A,C\}$: $(\mathring{x}_S, \mathring{y}_S) = (2,1)$.

INTERIOR S-ALLOCATIONS: PARETO FOR S REQUIRES

$MRS_A = MRS_C = 1$, BECAUSE $MRS_C = 1$ FOR ALL (x_C, y_C).

$\therefore MRS_A = 1$; $\therefore x_A = y_A$, AND $x_C = \mathring{x}_S - x_A = 2 - x_A$

$y_C = \mathring{y}_S - y_A = 1 - y_A = 1 - x_A$.

$\therefore u_A = \sqrt{x_A y_A} = \sqrt{x_A x_A} = x_A$ AND

$u_C = \frac{1}{2}[(2-x_A) + (1-x_A)] = \frac{1}{2}[3 - 2x_A] = \frac{3}{2} - x_A = \frac{3}{2} - u_A$

$\therefore \boxed{u_A + u_C = \frac{3}{2}}$ FOR INTERIOR S-ALLOCATIONS, $\boxed{u_A \leqq 1, u_C \geqq \frac{1}{2}}$.

BOUNDARY PARETO ALLOCATIONS FOR S ARE THE ONES

AT WHICH $y_A = 1$, $y_C = 0$, $1 \leqq x_A \leqq 2$, $x_C = 2 - x_A$.

IN THIS CASE WE HAVE $u_C = \frac{1}{2}x_C$; $x_C = 2u_C$;

$y_A = 1$, SO $u_A = \sqrt{x_A}$; $x_A = u_A^2$,

$x_A + x_C = 2$; i.e., $\boxed{u_A^2 + 2u_C = 2}$. FOR $u_A \geqq 1$, $u_C \leqq \frac{1}{2}$.

WE COULD ALSO WRITE THIS AS $u_C = 1 - \frac{1}{2}u_A^2$.

$S = \{B, C\}$: $(\overset{\circ}{x}_S, \overset{\circ}{y}_S) = (1, 2)$.

INTERIOR S-ALLOCATIONS: PARETO FOR S REQUIRES $x_B = y_B$. AND $u_B = x_B = y_B$

$\therefore$ $x_C = 1 - x_B$ AND $y_C = 2 - y_B = 2 - x_B$.

$\therefore$ $u_C = \frac{1}{2}\left[(1-x_B)+(2-x_B)\right] = \frac{1}{2}(3-2x_B) = \frac{3}{2} - x_B = \frac{3}{2} - u_B$;

i.e., $\boxed{u_B + u_C = \frac{3}{2}}$ FOR INTERIOR S-ALLOCATIONS; $\boxed{u_B \leqq 1, u_C \geqq \frac{1}{2}}$

BOUNDARY PARETO ALLOCATIONS FOR S: THERE ARE ONLY $(x_B, y_B) = (0,0)$ AND $(x_B, y_B) = (1,1)$; IN THE LATTER CASE, $(x_C, y_C) = (0, 1)$ AND $u_B = 1$, $u_C = \frac{1}{2}$.

THERE IS NO WAY TO INCREASE u_B, AND NO WAY TO INCREASE u_C WITHOUT DECREASING u_B, ACCORDING TO THE EQUATION $u_B + u_C = \frac{3}{2}$ ABOVE.

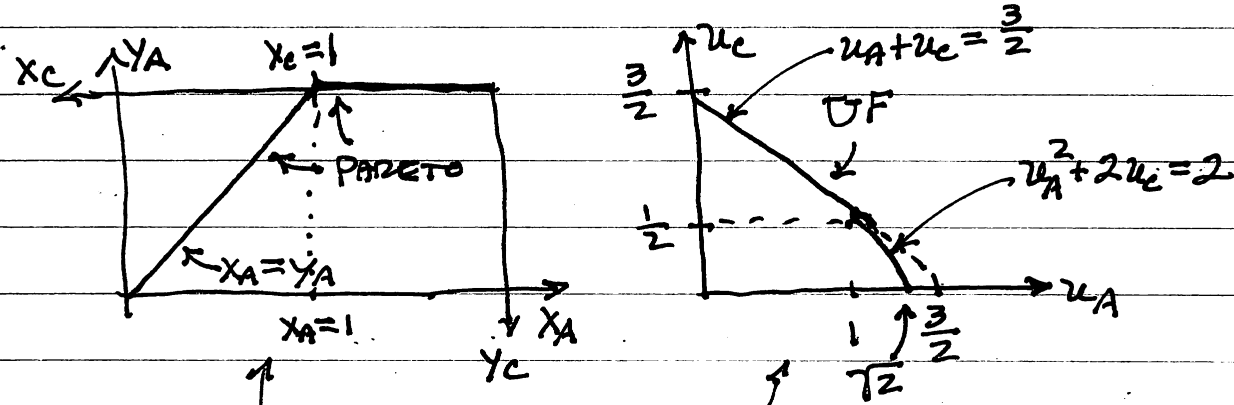


FIGURE 1: $S = \{A, C\}$.

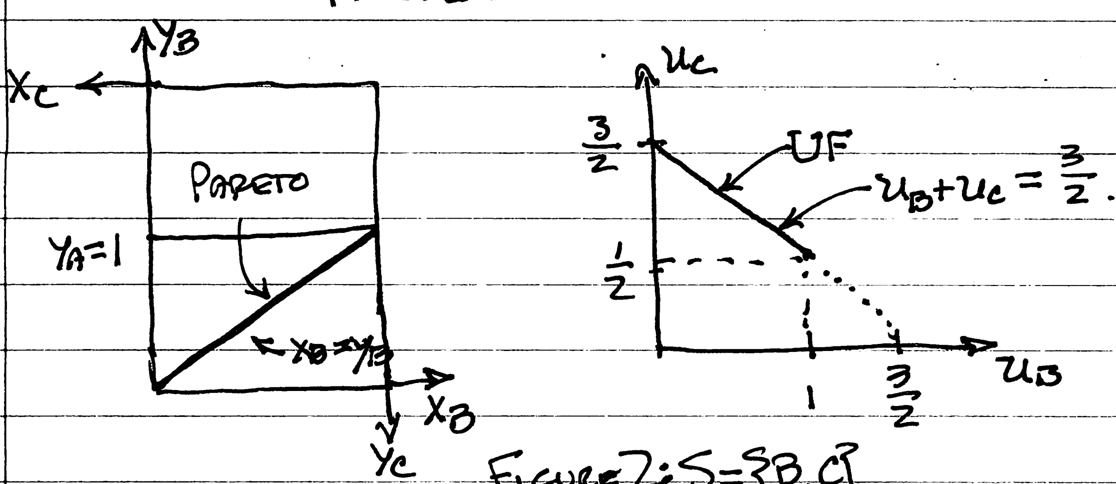


FIGURE 2: $S = \{B, C\}$

(8) CORE ALLOCATIONS ARE PARETO, THEREFORE $x_i = y_i$ FOR $i = A, B, C$. ALSO $x_A + x_B + x_C = \mathring{x} = 2$ AND $y_A + y_B + y_C = \mathring{y} = 2$.

RESTRICTIONS IMPOSED BY THE VARIOUS COALITIONS BEING UNABLE TO IMPROVE ON AN ALLOCATION:

(NOTE THAT SINCE $x_i = y_i$ ($\forall i$), WE HAVE
$u_A = x_A = y_A$, $u_B = x_B = y_B$, $u_C = \frac{1}{2}(x_C + y_C) = x_C = y_C$.)

$S = \{C\}$: $u_C \geqq \mathring{u}_C = \frac{1}{2}(1+1) = 1$; i.e., $x_C = y_C \geqq 1$.

$S = \{A, B\}$: $u_A + u_B \geqq 1$; i.e., $x_A + x_B = y_A + y_B \geqq 1$.

COMBINING THE ABOVE TWO RESTRICTIONS, WE ALREADY HAVE $x_C = y_C = 1$ AND $x_A + x_B = y_A + y_C = 1$.

$S = \{A, C\}$: $u_A + u_C \geqq \frac{3}{2}$; i.e., $x_A + x_C = y_A + y_C \geqq \frac{3}{2}$. $\therefore x_B = y_B \leqq \frac{1}{2}$

$S = \{B, C\}$: $u_B + u_C \geqq \frac{3}{2}$; i.e., $x_B + x_C = y_B + y_C \geqq \frac{3}{2}$. $\therefore x_A = y_A \leqq \frac{1}{2}$.

BUT $x_A + x_B = y_A + y_B = 1$; $\therefore x_A = y_A = x_B = y_B = \frac{1}{2}$.

$\therefore$ THE ONLY CORE ALLOCATION IS
$(x_A, y_A) = (x_B, y_B) = (\frac{1}{2}, \frac{1}{2})$ AND $(x_C, y_C) = (1, 1)$.