Economics 501B Fall 2014 Final Exam Solutions

1. The First Welfare Theorem: If $\left(\widehat{\mathbf{p}}, \left(\widehat{\mathbf{x}}^i\right)_1^n\right)$ is a Walrasian equilibrium for an economy $E = \left((u^i, \mathring{\mathbf{x}}^i)\right)_1^n$ in which each u^i is locally nonsatiated, then $(\widehat{\mathbf{x}}^i)_1^n$ is a Pareto allocation for E.

Proof:

Suppose $(\widehat{\mathbf{x}}^i)_1^n$ is not a Pareto allocation — *i.e.*, some allocation $(\widetilde{\mathbf{x}}^i)_1^n$ is a Pareto improvement on $(\widehat{\mathbf{x}}^i)_1^n$:

$$\begin{array}{ll}
(a) & \sum_1^n \widetilde{\mathbf{x}}^i \leqq \sum_1^n \mathring{\mathbf{x}}^i \\
(b1) & \forall i \in N: \ u^i(\widetilde{\mathbf{x}}^i) \geqq u^i(\widehat{\mathbf{x}}^i) \\
(b2) & \exists i' \in N: \ u^{i'}(\widetilde{\mathbf{x}}^{i'}) > u^{i'}(\widehat{\mathbf{x}}^{i'}).
\end{array}$$

Because $\left(\widehat{\mathbf{p}}, (\widehat{\mathbf{x}}^i)_1^n\right)$ is a Walrasian equilibrium for E, each $\widehat{\mathbf{x}}^i$ maximizes u^i on the budget set $\mathcal{B}(\widehat{\mathbf{p}}, \mathring{\mathbf{x}}^i) := \{ \mathbf{x}^i \in \mathbb{R}^l_+ \mid \widehat{\mathbf{p}} \cdot \mathbf{x}^i \leqq \widehat{\mathbf{p}} \cdot \mathring{\mathbf{x}}^i \}$. Therefore, (b2) implies that

$$\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^{i'} > \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^{i'}, \tag{1}$$

and since each u^i is locally nonsatiated, (b1) implies that

$$\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^{i} \geqq \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^{i} \quad \text{for each } i. \tag{2}$$

Note that (7) follows from the First Duality Theorem, which says that if $\succsim$ is a locally nonsatiated preference on a set X of consumption bundles in $\mathbb{R}^\ell_+$, and if $\widehat{\mathbf{x}}$ is $\succsim$-maximal in the budget set $\{x \in X \mid \mathbf{p} \cdot \mathbf{x} \leqq \mathbf{p} \cdot \widehat{\mathbf{x}}\}$, then $\widehat{\mathbf{x}}$ minimizes $\mathbf{p} \cdot \mathbf{x}$ over the upper-contour set $\{x \in X \mid \mathbf{x} \succsim \widehat{\mathbf{x}}\}$.

Summing the inequalities in (1) and (2) yields

$$\sum_{i=1}^n \widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i > \sum_{i=1}^n \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^i, \tag{3}$$

i.e.,

$$\widehat{\mathbf{p}} \cdot \sum_{i=1}^n \widetilde{\mathbf{x}}^i > \widehat{\mathbf{p}} \cdot \sum_{i=1}^n \widehat{\mathbf{x}}^i. \tag{4}$$

Since $\widehat{\mathbf{p}} \in \mathbb{R}^l_+$, it follows from (4) that there is at least one k for which

$$\widehat{p}_k > 0 \ \text{ and } \ \sum_{i=1}^n \widetilde{x}^i_k > \sum_{i=1}^n \widehat{x}^i_k. \tag{5}$$

Since $\widehat{p}_k > 0$, the market-clearing equilibrium condition yields $\sum_{i=1}^n \widehat{x}^i_k = \sum_{i=1}^n \mathring{x}^i_k$, and (5) therefore yields $\sum_{i=1}^n \widetilde{x}^i_k > \sum_{i=1}^n \mathring{x}^i_k$ — *i.e.*, $(\widetilde{\mathbf{x}}^i)_1^n$ does not satisfy (a). Our assumption that $(\widetilde{\mathbf{x}}^i)_1^n$ is a Pareto improvement has led to a contradiction; therefore there are no Pareto improvements on $(\widehat{\mathbf{x}}^i)_1^n$, and it's therefore a Pareto allocation. ■

(2) THIS JOINT PRODUCT IS LIKE A PUBLIC GOOD, AS IN ARROW'S MODEL IN OUR LECTURE NOTES.

(a) PARETO EFFICIENCY (INTERIOR) REQUIRES THAT $\sum MRS^i = MC$ — i.e., $24-3x=18$ HERE, SO WE HAVE $x=2$: $x_A=x_B=x_C=2$.

(b) LINDAHL PRICES WILL LEAD EACH PERSON TO CHOOSE THE SAME (PARETO) AMOUNT OF THE GOOD:

$p_A = MRS^A = \$8$, $p_B = MRS^B = \$6$, $p_C = MRS^C = \$4$ AT $x=2$.

THIS YIELDS REVENUE OF $(8+6+4)(2) = \$36$, EQUAL TO THE COST OF $x=2$. IT ALSO YIELDS THE PARETO AMOUNT OF THE GOOD, WHICH IS ONE NOTION OF MAXIMIZING WELFARE. AND IN THIS QUASILINEAR-UTILITY ENVIRONMET, A PARETO OUTCOME ALSO MAXIMIZES CONSUMER SURPLUS.

(c) IF $p<\$6$, THEN $x_i>0$ $(i=A,B,C)$ — i.e., ALL THREE CONSUMERS PURCHASE POSITIVE AMOUNTS OF ELECTRICITY — BUT PER-UNIT REVENUE IS LESS THAN $3p < 3(\$6) = \$18$, THE PER-UNIT COST. IF $\$6 \leq p < \$8$, WE HAVE $x_A, x_B > 0$, BUT PER-UNIT REVENUE IS LESS THAN $2p < 2(\$8) < \$18$, AGAIN LESS THAN COST. IF $\$8 \leq p < \$10$, ONLY $x_A > 0$, AND PER-UNIT REVENUE OF $\$10$ IS AGAIN LESS THAN COST. THEREFORE ALL PRICES THAT ELICIT ANY PURCHASES PRODUCE A LOSS FOR THE FIRM. SO ITS PROFIT IS MAXIMIZED BY NOT PRODUCING (PROFIT $= \$0$), WHICH IS ALSO CONSISTENT WITH THE FIRM CHARGING $p \geq \$10$.

(d) If the firm prices in such a way that, say, $x_A > x_B, x_C$, then on every unit that x_A exceeds x_B — i.e. on $x_A - x_B$ — the firm earns revenue from A only, at a price $p_A < \$10$ that doesn't cover the cost of those additional units. Similarly, if $x_A, x_B > x_C$. Therefore the firm will set prices p_A, p_B, p_C that elicit $x_A = x_B = x_C$. Denote this common demand by x.

(e) To elicit x from each consumer, the firm must charge

$$p_A = 10 - x, \quad p_B = 8 - x, \quad p_C = 6 - x,$$

and its revenue will be

$$R(x) = (10-x)x + (8-x)x + (6-x)x = 24x - 3x^2.$$

Therefore $MR = 24 - 6x$ and $R(\cdot)$ is strictly concave; profit-maximization requires that $MR = MC$ — i.e. $24 - 6x = 18$, so $x = 1$.

Or you can just maximize $R(x) - C(x) = 24x - 3x^2 - 18x$.

The firm charges $p_A = \$9$, $p_B = \$7$, $p_C = \$5$, so the firm's revenue is \$21 and its cost is \$18, for a profit of \$3. Note that this is not the Pareto quantity of x.

Note that $CS_A = \frac{1}{2}(1)(\$1) = \$\frac{1}{2} = CS_B = CS_C$, so total CS is $\$\frac{3}{2}$ and total surplus is $CS + \pi = \$\frac{3}{2} + \$3 = \$4\frac{1}{2}$.

At the Lindahl allocation we have

$$CS_A = \tfrac{1}{2}(2)(\$2) = \$2 = CS_B = CS_C, \text{ so total } CS = \$6,$$

which is total surplus, because $\pi = \$0$.

(3) $u^i(x,y) = \sqrt{x} + \sqrt{y}$, $MRS = \dfrac{\sqrt{y}}{\sqrt{x}}$.

$(\mathring{x}_A, \mathring{y}_A) = (0,4)$, $(\mathring{x}_B, \mathring{y}_B) = (4,0)$.

(a) AT PARETO ALLOCATIONS WE HAVE $MRS^A = MRS^B$ — i.e., $\dfrac{\sqrt{y_A}}{\sqrt{x_A}} = \dfrac{\sqrt{y_B}}{\sqrt{x_B}} = r$, SAY. THEREFORE

$\sqrt{y_i} = r\sqrt{x_i}$ $(i = A, B)$, AND

$$u_i = \sqrt{x_i} + \sqrt{y_i} = \sqrt{x_i} + r\sqrt{x_i} = (1+r)\sqrt{x_i},$$

AND $u_i^2 = (1+r)^2 x_i$, SO

$$u_A^2 + u_B^2 = (1+r)^2(x_A + x_B) = (1+r)^2 \mathring{x}.$$

SIMILARLY, $u_i = \sqrt{x_i} + \sqrt{y_i} = \dfrac{1}{r}\sqrt{y_i} + \sqrt{y_i} = \dfrac{1+r}{r}\sqrt{y_i}$,

AND $u_i^2 = \dfrac{(1+r)^2}{r^2} y_i$, SO

$$u_A^2 + u_B^2 = \frac{(1+r)^2}{r^2}(y_A + y_B) = \frac{(1+r)^2}{r^2}\mathring{y}.$$

THEREFORE WE HAVE $\dfrac{(1+r)^2}{r^2}\mathring{y} = (1+r)^2 \mathring{x}$ — i.e., $r^2 = \dfrac{\mathring{y}}{\mathring{x}}$,

AND $r = \dfrac{\sqrt{\mathring{y}}}{\sqrt{\mathring{x}}}$. $\therefore u_A^2 + u_B^2 = \left(1 + \dfrac{\sqrt{\mathring{y}}}{\sqrt{\mathring{x}}}\right)^2 \mathring{x}$.

(b) SINCE WE HAVE $\mathring{x} = \mathring{y} = 4$, THE PARETO FRONTIER IS $u_A^2 + u_B^2 = (1+1)^2 \mathring{x} = (4)(4) = 16$.

(b) ONE-PERSON COALITIONS ("INDIVIDUAL RATIONALITY"):

$$\mathring{u}_A = \sqrt{\mathring{x}_A} + \sqrt{\mathring{y}_A} = \sqrt{0} + \sqrt{4} = 2$$

$$\mathring{u}_B = \sqrt{\mathring{x}_B} + \sqrt{\mathring{y}_B} = \sqrt{4} + \sqrt{0} = 2$$

$\therefore$ WE MUST HAVE $u_A \geqq 2$ AND $u_B \geqq 2$ FOR A CORE ALLOCATION.

TWO-PERSON COALITION (PARETO):

$MRS^A = MRS^B$; i.e., $\sqrt{\dfrac{y_A}{x_A}} = \sqrt{\dfrac{y_B}{x_B}}$;

i.e., $\dfrac{y_A}{x_A} = \dfrac{y_B}{x_B} = \dfrac{4 - y_A}{4 - x_A}$;

i.e., $4y_A - x_A y_A = 4x_A - x_A y_A$

i.e., $y_A = x_A$; $\therefore y_B = x_B$.

$\therefore u_A = \sqrt{x_A} + \sqrt{x_A} = 2\sqrt{x_A}$ AND $u_B = 2\sqrt{x_B}$.

COMBINING:

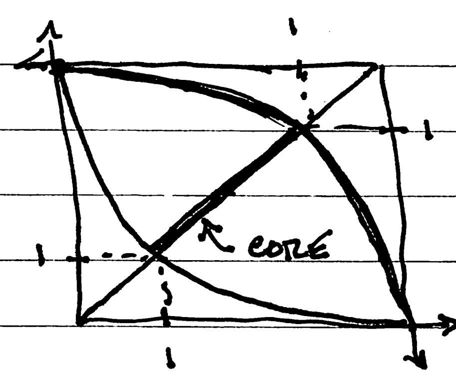
$2\sqrt{x_A} \geqq 2$; i.e., $x_A \geqq 1$

$2\sqrt{x_B} \geqq 2$; i.e., $x_B \geqq 1$.

$\therefore$ THE CORE ALLOCATIONS ARE THE ONES THAT SATISFY

$y_A = x_A \geqq 1$, $y_B = x_B \geqq 1$

$x_A + y_A = x_B + y_B = 4$



(4) $MRS_H^A = 1 - \frac{1}{30}x_H^A$, $MRS_L^A = 1 - \frac{1}{10}x_L^A$ $\quad \mathring{x}^A = (30, 5, 25)$

$MRS_H^B = 1 - \frac{1}{30}x_H^B$, $MRS_L^B = 3 - \frac{1}{10}x_L^B$ $\quad \mathring{x}^B = (30, 35, 5)$

(a) $MRS_H^A = MRS_H^B$: $1 - \frac{1}{30}x_H^A = 1 - \frac{1}{30}x_H^B$; $\therefore$ $\boxed{x_H^A = x_H^B = 20.}$

$MRS_L^A = MRS_L^B$: $1 - \frac{1}{10}x_L^A = 3 - \frac{1}{10}x_L^B$; $\therefore$ $x_L^B - x_L^A = 20$; $\boxed{x_L^A = 5,\ x_L^B = 25.}$

$\boxed{x_0^A + x_0^B = 60.}$

(b) THE EQUILIBRIUM IS PARETO (1ST WELFARE THEOREM), SO $p_\theta = MRS_\theta^i$ $(\forall i, \theta)$ AT THE ALLOCATION IN (a):

$MRS_H^i = 1 - \frac{2}{3} = \frac{1}{3}$ $(\forall i)$ AND $MRS_L^i = 1 - \frac{1}{2} = 3 - \frac{5}{2} = \frac{1}{2}$ $(\forall i)$,

SO $\boxed{p_H = \frac{1}{3} \text{ AND } p_L = \frac{1}{2}.}$

$x_0^A = 30 - \frac{1}{3}(20-5) - \frac{1}{2}(5-25) = 30 - 5 + 10 = 35$

$x_0^B = 30 - \frac{1}{3}(20-35) - \frac{1}{2}(25-5) = 30 + 5 - 10 = 25.$

$\therefore$ $\boxed{x^A = (35, 20, 5) \text{ AND } x^B = (25, 20, 25).}$

(c) THE SECURITIES RETURNS MATRIX IS $D = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1+r & 0 \\ 1+r & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ H \\ L \end{array}.$

SECURITIES PRICES:

$q_1 = (1+r)p_H + (1+r)p_L = (1+r)\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5}{6}(1+r) = 1$; $\therefore$ $\boxed{r = \frac{1}{5} = 20\%.}$

$\boxed{q_2 = 0p_H + 1p_L = p_L = \frac{1}{2}.}$ [SEE * BELOW]

SINCE THE INSURANCE CONTRACT RETURNS ZERO IN STATE H, ALL CONTRACTING FOR STATE H MUST BE DONE VIA THE BOND. SINCE $x_H^A - \mathring{x}_H^A = 15$ AND $x_H^B - \mathring{x}_H^B = -15$, AND $1+r = \frac{6}{5}$,

A WILL ~~SELL~~ BUY $\frac{5}{6}(15) = 12\frac{1}{2}$ UNITS: $\boxed{y_1^A = 12\frac{1}{2}}$

B WILL ~~BUY~~ SELL $\frac{5}{6}(15) = 12\frac{1}{2}$ UNITS: $\boxed{y_1^B = -12\frac{1}{2}.}$

THE BOND CONTRACT COVERS EVERYONE IN STATE H; IT ALSO PAYS OFF IN STATE L, BUT EACH CONSUMER ALSO NEEDS

THE STATE-L RETURN FROM THE INSURANCE CONTRACT TO MAKE UP THE DIFFERENCE BETWEEN x_L^i AND $\overset{\circ}{x}_L^i + (1+r)y_1^i$:

$$y_2^A = x_L^A - [\overset{\circ}{x}_L^A + (1+r)y_1^A] = 5 - [25+15] = 5-40 = \boxed{-35}$$
$$y_2^B = x_L^B - [\overset{\circ}{x}_L^B + (1+r)y_1^B] = 25 - [5-15] = 25+10 = \boxed{35}.$$

(d) A PARETO ALLOCATION WILL GENERALLY HAVE STATE-DEPENDENT NET CONSUMPTION INCREMENTS $x_\theta^i - \overset{\circ}{x}_\theta^i$ THAT DIFFER FOR A CONSUMER, BUT WITH JUST THE CREDIT MARKET THESE ~~IS~~ INCREMENTS ARE THE SAME IN EACH STATE. MORE GENERALLY, WITH JUST ONE SECURITY THE RETURNS ARE COLINEAR. A SECOND SECURITY, WITH RETURNS LINEARLY INDPENDENT FROM THE FIRST SECURITY, ENABLES THE CONSUMER TO INDEPENDENTLY VARY HIS RETURNS IN THE TWO STATES, AS THE INSURANCE MARKET MADE POSSIBLE IN (c).

* IF THIS IS MODELED WITH A RETURNS MATRIX $D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, THEN WE GET $q_1 = \frac{5}{6} = \frac{1}{1+r}$, SO AGAIN $r = \frac{1}{5}$, AND WE GET $y_1^A = 15$ AND $y_1^B = -15$.

EVERYTHING ELSE IS THE SAME.

NOTE THAT (EITHER WAY)

$$x_0^A - \overset{\circ}{x}_0^A = q_1 y_1^A + q_2 y_2^A = 12\tfrac{1}{2} - 17\tfrac{1}{2} = -5$$
$$x_0^B - \overset{\circ}{x}_0^B = q_1 y_1^B + q_2 y_2^B = -12\tfrac{1}{2} + 17\tfrac{1}{2} = 5.$$