Economics 501B Midterm Exam
University of Arizona
Fall 2017

1. Assume that a bundle $\mathring{\mathbf{x}} = (\mathring{x}_1, \ldots, \mathring{x}_\ell) \in \mathbb{R}^\ell_+$ of ℓ goods is to be allocated among n individuals, whose preferences over allocations $(\mathbf{x}^i)_1^n \in \mathbb{R}^{n\ell}_+$ are described by "selfish" utility functions $u^i(\cdot) : \mathbb{R}^\ell_+ \to \mathbb{R}$, $i = 1, \ldots, n$ — *i.e.*, each $u^i(\cdot)$ depends only on person i's bundle $\mathbf{x}^i$. Let N denote the set $\{1, \ldots, n\}$; let $(\widehat{\mathbf{x}}^i)_N$ be some particular allocation; and for each $i \in N$, let $\widehat{u}_i = u^i(\widehat{\mathbf{x}}^i)$.

For each $h \in N$ define the maximization problem P-Max[h] as follows:

$$\max_{(\mathbf{x}^i) \in \mathbb{R}^{n\ell}_+} u^h(\mathbf{x}^h) \quad \text{subject to}$$

$$\sum_{i=1}^n x^i_k \leqq \mathring{x}_k, \quad k = 1, ..., l$$

$$u^i(\mathbf{x}^i) \geqq \widehat{u}_i, \quad i \neq h.$$

Note that this is a collection of n different maximization problems, each with a different person's utility function used as the objective function and no utility-level constraint for that person.

In Econ 501B we have proved that

(a) if $(\widehat{\mathbf{x}}^i)_N$ is a Pareto allocation, then it is a solution of P-Max[1], (and obviously, it's also a solution of P-Max[h] for every $h \in N$), and

(b) if each $u^i(\cdot)$ is continuous and locally nonsatiated, and $(\widehat{\mathbf{x}}^i)_N$ is a solution of P-Max[h] for some $h \in N$, then $(\widehat{\mathbf{x}}^i)_N$ is Pareto optimal.

The well-known economist Risheng Xu has proposed the following conjecture as an alternative to (b):

Xu's Conjecture: If $(\widehat{\mathbf{x}}^i)_N$ is a solution of P-Max[h] for *every* $h \in N$, then $(\widehat{\mathbf{x}}^i)_N$ is Pareto optimal. (Note that the conjecture does not assume that the utility functions are continuous or locally nonsatiated.)

Determine whether Xu's Conjecture is true or false, and verify your answer (with either a proof or a counterexample).

2. Abby's and Beth's preferences are both described by the utility function $u(x, y) = xy$. Abby owns the bundle (1,9) and Beth owns the bundle (9,1); the utility profile at this initial allocation is therefore $(\mathring{u}_A, \mathring{u}_B) = (9, 9)$.

(a) Determine the set of Pareto allocations and depict this set in an Edgeworth box diagram. (You don't need to derive the Pareto allocations from first principles, but show how you determined the Pareto allocations.)

(b) Determine the Walrasian equilibrium prices and allocation. You needn't do this by deriving the equilibrium, but you should verify that what you have is an equilibrium (*i.e.*, that it satisfies the equilibrium conditions). Indicate this allocation in your diagram for part (a). What is the utility profile (u_A, u_B) at the Walrasian equilibrium?

(c) Determine the set of core allocations and the set of utility profiles (u_A, u_B) attainable via core allocations. Depict the core in your diagram for part (a).

Now suppose that Abby and Beth are joined by a third person, Cathy, who has the same preference as the others, but who owns the bundle (5,5).

(d) Determine the set of Pareto allocations.

(e) Determine the Walrasian equilibrium prices and allocation and the utility profile (u_A, u_B, u_C) at the Walrasian allocation.

(f) Determine the core allocations. For this you might find it helpful to know that $\sqrt{84}$ is approximately 9.17.

(g) In both the Walrasian equilibrium and the core Cathy does no better (or worse) in the presence of Abby and Beth than if she were alone. But the bundles and utility profiles attainable by Abby and Beth in the core (*i.e.*, as bargaining equilibria) are affected by the presence of Cathy. Why?

3. There are two goods (denote their quantities by x and y) and two consumers, Amy and Bev. No production is possible. Each consumer owns the same bundle of goods, $(\mathring{x}_i, \mathring{y}_i) = (9, 16)$, and each has the same preference, which is representable by the utility function $u(x, y) = \sqrt{x} + \sqrt{y}$. Note that the MRS function for each consumer is $\sqrt{y/x}$.

(a) Determine the Walrasian equilibrium prices and allocation.

(b) The allocation in (a) is Pareto optimal. Is it still Pareto optimal if the initial allocation is $(\overline{x}_A, \overline{y}_A) = (0, 32)$ for Amy and $(\overline{x}_B, \overline{y}_B) = (18, 0)$ for Bev? Explain. The Second Welfare Theorem says that Pareto optimal allocations can be supported as Walrasian equilibria. If the allocation in (a) *is* Pareto optimal for the initial allocation $\big((\overline{x}_A, \overline{y}_A), (\overline{x}_B, \overline{y}_B)\big)$, is it a Walrasian equilibrium for this initial allocation? If the allocation in (a) is not Pareto optimal for this initial allocation, find a Pareto improvement.

(c) Determine all the initial allocations for which the allocation in (a) is a Walrasian equilibrium for some prices.

(d) Production has become possible! And now Amy and Bev each initially own the bundle $(9, 18)$. There are two firms: Amy owns one firm and Bev owns the other firm. Each firm's production function is $q = f(z) = 2z$, where z is the amount of input used (the x-good) and q is the amount of output produced (the y-good). Determine the Walrasian equilibrium — the prices, the bundles consumed by each consumer, and the production plans and profits of the firms.