Economics 501B Final Exam
University of Arizona
Fall 2015

1. Ann, Bev, and Cat are the only residents on the shore of a small lake. Each woman's preference for the water level in the lake is represented by a differentiable strictly quasiconcave utility function $u^i(x, y_i)$, where x denotes the water level and y_i denotes the dollars Ms. i will have available to spend each week. The water level can be raised or lowered at no cost.

(a) Derive the marginal condition that characterizes the Pareto efficient interior allocations (x, y_1, y_2, y_3).

(b) The women use the following arrangement to decide on the level at which they will set the water each week: Each woman submits a vote, a nonnegative number m_i; the water level is set at the mean of the three votes: $x = \frac{1}{3}(m_A + m_B + m_C)$; and they pay the following amounts, t_i:

$$t_A = (m_B - m_C)x, \quad t_B = (m_C - m_A)x, \quad t_C = (m_A - m_B)x.$$

Note that a payment t_i can be either positive or negative (in which case it is an amount received), or of course can be zero. Determine whether an interior Nash equilibrium will be Pareto efficient.

For the remainder of this problem assume that the utility functions are all of the form $u^i(x, y_i) = y_i + a_i x - \frac{1}{2}x^2$ and that $a_A = 4, a_B = 1, a_C = 1$, and assume that each woman has \$100 each week.

(c) Determine the Pareto efficient interior allocations.

(d) For each i let β_i be Ms. i's "best" water level, *i.e.*, the water level she most prefers. Determine β_A, β_B, and β_C. Under the arrangement in (b), if each woman votes for her most-preferred water level — *i.e.*, if $\forall i : m_i = \beta_i$ — will the outcome be Pareto efficient? Is this a Nash equilibrium? Verify your answers.

(e) Are there any interior or boundary Pareto allocations in which $x = 1$? If so, identify one such allocation; if not, explain why not.

(f) Determine all the Nash equilibria: for each equilibrium determine each woman's vote, the water level, and the amount each woman pays or receives.

2. Consider the problem of allocating two goods among n persons, each of whom has the same preference over the two goods, which is represented by the utility function $u(x, y) = \sqrt{x} + \sqrt{y}$. Note that the MRS function for this utility function is $MRS = \sqrt{y}/\sqrt{x}$. **Note:** Each of the parts of this problem can be solved independently of the other parts. For example, you'll probably find the equation in (a) useful for some of the other parts, but those parts don't depend on your being able to verify the equation; it won't matter for the other parts if you're unable to determine a maximization problem as requested in (c); etc.

(a) Verify that the utility frontier is given by the equation $\sum_1^n u_i^2 = (\sqrt{\mathring{x}} + \sqrt{\mathring{y}}\,)^2$, where $\mathring{x}$ and $\mathring{y}$ are the total amounts of the two goods available to be allocated. Draw a diagram of the utility frontier for the case $n = 2$ and $(\mathring{x}, \mathring{y}) = (16, 9)$.

(b) For the case $n = 2$ and the utility function given above for each person, prove that an allocation $\big((x_1, y_1), (x_2, y_2)\big)$ is Pareto efficient if and only if there are nonnegative weights α_1 and α_2, not both zero, such that $\big((x_1, y_1), (x_2, y_2)\big)$ maximizes the social welfare function

$$W\big((x_1, y_1), (x_2, y_2)\big) = \alpha_1 u(x_1, y_1) + \alpha_2 u(x_2, y_2).$$

subject to the feasibility constraint $(x_1, y_1) + (x_2, y_2) \leqq (\mathring{x}, \mathring{y})$.

(c) For the case $n = 2$ the equation in (a) expresses u_1 as an implicit function of u_2, as your diagram in (a) presumably makes clear. We can rewrite the equation so that the function is explicit: $u_1 = v(u_2) = \sqrt{(\sqrt{\mathring{x}} + \sqrt{\mathring{y}}\,)^2 - u_2^2}$. Write down a constrained maximization problem for which $v(\cdot)$ is the value function. Your maximization problem will also have a solution function; describe what the solution function tells you. (You're not being asked to prove anything here, just to write down an appropriate maximization problem and explain its solution function.)

(d) Assume that $n = 2$ and that $(\mathring{x}_1, \mathring{y}_1) = (16, 0)$ and $(\mathring{x}_2, \mathring{y}_2) = (0, 16)$. Determine all the Walrasian equilibria (the equilibrium price-lists and allocations), all the Pareto efficient allocations, and all the core allocations. Depict all three sets of allocations in an Edgeworth box diagram.

For the remainder of Problem #2 assume that $n = 3$ and that

$$(\mathring{x}_1, \mathring{y}_1) = (16, 0), \qquad (\mathring{x}_2, \mathring{y}_2) = (0, 16), \qquad (\mathring{x}_3, \mathring{y}_3) = (9, 9).$$

(e) Determine all Walrasian equilibria.

(f) Are there any core allocations in which $(x_3, y_3) \neq (9, 9)$? If so, identify one such allocation; if not, verify that.

(g) Show that any core allocation in which $(x_3, y_3) = (9, 9)$ must satisfy $x_i, y_i \geqq 7$ and $x_i, y_i \leqq 9$ for $i = 1, 2$.

3. Bart and Arnie each have income today of $12K per month. If the Democrats win the next election Bart's income will be $10K per month; if the Republicans win, his income will be $30K per month. Arnie's income will be $15K per month whichever party wins the election. Arnie's and Bart's preferences are described by the utility functions

$$u^A(x_0, x_D, x_R) = x_0 + 3\sqrt{x_D} + 2\sqrt{x_R} \qquad \text{and} \qquad u^B(x_0, x_D, x_R) = x_0 + 4\sqrt{x_D} + \sqrt{x_R},$$

where x_0 denotes the individual's monthly spending today, x_D denotes his monthly spending tomorrow if the Democrats win, and x_R denotes his monthly spending tomorrow if the Republicans win (all expressed in thousands of dollars — for example, $x_R = 16$ represents spending of $16K per month if the Republicans win). Note that Arnie's and Bart's marginal rates of substitution are

$$MRS_D^A = \frac{3}{2\sqrt{x_D^A}}, \quad MRS_R^A = \frac{1}{\sqrt{x_R^A}}, \quad MRS_D^B = \frac{2}{\sqrt{x_D^B}}, \quad MRS_R^B = \frac{1}{2\sqrt{x_R^B}}.$$

(a) Determine the interior Pareto efficient allocation(s) by using the marginal conditions that characterize the Pareto allocations. (You needn't derive the marginal conditions from the Pareto definition or from the associated Pareto maximization problem.)

(b) Determine the Arrow-Debreu allocation(s) and prices.

When possible in the following parts, you can appeal to Arrow's securities pricing formula.

(c) In the Arrow-Debreu "contingent claims" market structure, what is the (implicit) interest rate?

(d) Suppose there are markets for two securities: one is a bond that pays $1K per month tomorrow independently of who wins the election; the other security pays $3K per month if the Democrats win and $2K per month if the Republicans win. Will both of these securities trade in equilibrium, or will one dominate the other so that only the dominant security trades? Explain. Determine the equilibrium prices of the securities, how much each person will hold of each security, and what each person's spending stream will be. What will be the interest rate?

(e) Can either of the two utility functions u^A or u^B be represented as expected von Neumann-Morgenstern utility over the outcomes x_D and x_R? If so, show how; if not, show why not.