Lindahl Equilibrium

Suppose that five homeowners live on the shore of Lake Magnavista: Amy, Bev, Cat, Dee, and Eve. In order to deal with such public goods problems as deciding on the water level in the lake and how to control mosquitoes in the summer, they've formed a homeowners' association (HOA for short).

Concerning the mosquitoes, the five women's preferences are all described by utility functions of the form $u(x, y_i) = y_i - \frac{1}{2}(\alpha_i - x)^2$, where x denotes the number of tankfuls of mosquito spray that are sprayed each week during the summer, and y_i denotes the amount of money homeowner i has available to spend on other goods. The values of their preference parameters α_i are

$$\alpha_A = 30, \qquad \alpha_B = 27, \qquad \alpha_C = 24, \qquad \alpha_D = 21, \qquad \alpha_E = 18,$$

and their MRS functions are therefore

$$MRS_A = 30 - x, \quad MRS_B = 27 - x, \quad MRS_C = 24 - x, \quad MRS_D = 21 - x, \quad MRS_E = 18 - x.$$

There are several local firms that will spray to control mosquitoes. The firms all charge the same price, $p = \$40$ per tankful they spray. This \$40 is therefore the marginal cost to the homeowners of a tankful of bug spray. Because of the homeowners' quasilinear utility functions, there is a unique Pareto amount of bug spray for them, namely $x = 16$ tanks: $\Sigma MRS_i = 120 - 5x$ and $MC = 40$, so $\Sigma MRS_i = MC$ at $x = 16$.

If the homeowners each contract separately with bug-spray firms to spray, each taking the others' purchases as given, then none of them will purchase any spray at all: the \$40 cost for each unit exceeds everyone's MRS. And as we've seen, if the HOA instead creates a fund into which they all voluntarily contribute, and uses the contributed funds to purchase the mosquito spray, the same outcome will occur: no contributions will be forthcoming, and therefore no spray will be purchased.

Now let's note that the homeowners' marginal rates of substitution *at the Pareto amount of spray* would be

$$MRS_A = 14, \qquad MRS_B = 11, \qquad MRS_C = 8, \qquad MRS_D = 5, \qquad MRS_E = 2.$$

Suppose the homeowners decide that instead of each of them purchasing bug spray separately and each paying \$40 per tankful (resulting in no spray at all being purchased), their HOA will instead charge each of them only a *share* of the \$40 price: homeowner i will pay the price-share (or per-unit tax) p_i for each unit the HOA purchases, with $\sum_{i=1}^{5} p_i = 40$.

Suppose the HOA sets these price-shares in such a way that each person's share p_i is equal to her marginal value for bug spray — *i.e.*, her MRS — at the Pareto amount $x = 16$. Then

$$p_A = 14, \qquad p_B = 11, \qquad p_C = 8, \qquad p_D = 5, \qquad p_E = 2.$$

Now the HOA asks each homeowner "How much spray *in total* do you want the HOA to purchase, knowing you will pay your price-share p_i for each tankful that's sprayed?" If each homeowner behaves as a price-taker — taking her price-share p_i as given — how much spray will she request? Choosing (x, y_i) to maximize her utility subject to the budget constraint $p_i x + y_i = \mathring{y}_i$, each homeowner will choose the x at which $MRS_i = p_i$ — *i.e.*, each homeowner will request $x = 16$.

How much money will the HOA have available to pay for the 16 tanks of spray? It will collect $p_A + p_B + p_C + p_D + p_E = \$40$ for each tank that's sprayed — *i.e.*, exactly the marginal cost to the HOA of the spray. An alternative approach would be for the HOA to draw up an agreement with one of the firms, say Bug Spray, Inc. (BSI), as follows: BSI will charge different prices p_i to each homeowner and ask each homeowner to report how much spray, in total, she would like BSI to spray at that price; and BSI is to adjust the personal prices p_i until all the homeowners are in agreement — *i.e.*, until each homeowner requests the same amount of spray. This has all the earmarks of an *equilibrium*: personal prices and the amount produced and consumed are adjusted as long as the participants don't agree on that amount; and when the participants *do* agree, adjustments no longer occur.

This idea is due to the Swedish economist Erik Lindahl, who proposed it in 1919. Here is a formal definition of **Lindahl equilibrium** for the one-public-good, one-private-good case. It's straightforward to write down the definition for multiple public and private goods as well, but that requires more notation than I want to introduce here. We assume here that there is one public good (quantity denoted by x) and one "regular" or private good (with y_i denoting the quantity consumed by i). There are n consumers, with utility functions $u_i(x, y_i)$. There are m firms; each firm has a production function f_j according to which z_j units of input (the private good) are converted into $q_j = f_j(z_j)$ units of the public good. Each consumer i owns the share $\theta_{ij} \geqq 0$ of firm j's profit, and $\Sigma_i \theta_{ij} = 1$ for each $j = 1, \ldots, m$. Denote the price of the private good by p_y. There are **Lindahl prices** (also called **Lindahl taxes**) $p_1, \ldots, p_n$ that the consumers $i = 1, \ldots, n$ are charged for the public good.

Definition: For an economy as described above, a **Lindahl equilibrium** is

a price-list $(p_y^*, p_1^*, \ldots, p_n^*)$,

a production allocation $(z_1^*, \ldots, z_m^*)$, and

a consumption allocation $(x^*, y_1^*, \ldots, y_n^*)$

that satisfy the following conditions, where $p_x^* = \Sigma_1^n p_i^*$:

(π-Max) $\quad \forall j : z_j^*$ maximizes firm j's profit, $\pi_j(z_j) := p_x^* f_j(z_j) - p_y^* z_j$,

(U-Max) $\quad \forall i : (x^*, y_i^*)$ maximizes $u_i(x, y_i)$ subject to $p_i^* x + y_i \leqq \mathring{y}_i + \Sigma_{j=1}^m \theta_{ij}\pi_j(z_j^*)$,

(M-Clr-x) $\quad x^* \leqq \Sigma_1^m q_j^* \quad$ and $\quad x^* = \Sigma_1^m q_j^*$ if $p_x^* > 0$, where $q_j^* = f_j(z_j^*)$, $j = 1, \ldots, m$,

(M-Clr-y) $\quad \Sigma_1^n y_i^* + \Sigma_1^m z_j^* \leqq \Sigma_1^n \mathring{y}_i \quad$ and $\quad \Sigma_1^n y_i^* + \Sigma_1^m z_j^* = \Sigma_1^n \mathring{y}_i$ if $p_y^* > 0$.

Note that this has a certain parallel with the no-externalities Walrasian equilibrium: at both the Walrasian and Lindahl equilibria, the price that a consumer pays for a good is the same for every unit she consumes, and (if she is maximizing utility) the price is equal to her marginal rate of substitution, *i.e.*, her marginal value for the good. But in the Walrasian case everyone pays the *same* price, p, while here everyone will typically be paying a *different* price p_i. The Walrasian equilibrium definition implicitly assumes that the price will adjust if net demands don't sum to zero; the Lindahl equilibrium definition implicitly assumes that the price-shares and quantity will adjust if demands for the public good aren't all the same.

Samuelson argued, colorfully but informally, that the Lindahl idea is unworkable because it's unrealistic to expect people to take their Lindahl prices as given. (See, for example, "The Pure Theory of Public Expenditure", *Review of Economics and Statistics*, 1954.) Arrow, as we will see, provided a clearer, more formal version of this argument, but he drew a less sweeping conclusion from it than Samuelson had done.

The Lindahl equilibrium is useful because it provides a benchmark in which, just as in the Walrasian equilibrium, each consumer's per-unit payment to finance the public good is equal to his marginal value for the good, and no consumer is worse off at the equilibrium than if he instead just consumed his initial endowment, and the resulting allocation is Pareto optimal. These properties have motivated the design of game forms ("institutions") in which the Nash equilibrium actually yields Lindahl prices and a Lindahl allocation.