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ABSTRACT

Logical phi-bits are nonlinear acoustic modes analogous to qubits and supported by an externally driven acoustic metastructure. A correspondence is established between the state of three correlated logical phi-bits represented in a low-dimensional linearly scaling physical space and their state representation as a complex vector in a high-dimensional exponentially scaling Hilbert space. We show the experimental implementation of a nontrivial three phi-bit unitary operation analogous to a quantum circuit. This three phi-bit gate operates in parallel on the components of the three phi-bit complex state vector. While this operation would be challenging to perform in one step on a quantum computer, by comparison, ours requires only a single physical action on the metastructure.

Quantum computing[1,2] harnesses the quantum mechanical phenomena of superposition and entanglement. The first phenomenon provides the support for encoding massive amounts of information in coherent superpositions of states of a composite quantum system constituted of subsystems. The second phenomenon provides the correlation between the subsystems and the capability of processing information in a parallel manner. Quantum circuits model quantum computations as sequences of quantum gates operating on quantum bits (qubits). Qubits can be thought of as the subsystems of the computer composite system. A qubit is a two-level subsystem, with a superposition that is expressed in the two-dimensional basis corresponding to its two available pure states. Quantum gates are unitary operators described as unitary matrices acting on the state of the quantum system of interest expressed in some basis. For example, a single qubit gate operates on the qubit superposition expressed in its two-dimensional basis. Likewise, a two qubit gate operates on the superposition of states of the two qubits expressed on a $2^2 = 4$-dimensional basis, a tensor product of the individual qubit bases. Three qubit gates operate on states expressed on a $2^3 = 8$-dimensional basis. An N-bit gate operates on states supported by an exponentially scaling basis of dimension 2^N. It is this exponential scaling that potentially gives the advantage of quantum computing over classical computers. However, quantum computing faces the challenge of physically operating simultaneously on many qubit states while maintaining the quantum correlation between qubits during these operations. Quantum circuits are, therefore, decomposed into sequences of single or two qubit gates that can form a universal set of gates.

The phenomenon of coherent superposition of states is not limited to quantum systems, and classical waves, such as acoustic waves, can be experimentally prepared in coherent superpositions.[3–6] In linear acoustics, these superpositions involve bases that are related to the degrees of freedom associated with the medium supporting the waves.[3,4] The dimension of the basis is, therefore, limited to the number of degrees of freedom of the acoustic physical system. In nonlinear acoustics, this restriction can be lifted by visualizing the state of the system as a composite of the states associated with the unrestricted number of nonlinear acoustic modes.[5,6] Furthermore, while entanglement is necessary for the correlation of quantum systems, nonlinear acoustic coupling provides the correlation between nonlinear acoustic modes necessary for manipulating their superpositions simultaneously.[5] Quantum entanglement possesses two attributes: nonseparability of the wave function of a composite system into tensor product of wave functions of subsystems, and nonlocality. Classical entanglement of acoustic waves is achieved when the acoustic field or the corresponding wave function is not factorizable. Acoustic wave entanglement increases the number of possible states, and therefore, the range of information can be encoded and subsequently processed in those states. A final difference between acoustic waves and quantum waves is that classical waves represent amplitudes, and quantum waves are probability amplitudes. Therefore, classical waves do not suffer from wave function collapse of a superposition into a pure state upon a measurement. Coherent superpositions of acoustic waves are directly measurable. In contrast, the determination of a quantum superposition of states necessitates several measurements to account for the probabilistic nature of the quantum wave function.

In Ref. 5, we introduced the notion of logical phi-bit, a classical analogue of a qubit using nonlinear acoustic waves. Here, we demonstrate theoretically and experimentally that navigation among the states of three correlated phi-bits in their space of states, i.e., their Hilbert space, can achieve a nontrivial unitary operation analogous to a quantum gate. We show a correspondence between a low-dimensional linearly scaling space of physical parameters that control the coherent superposition of states of the three phi-bits in their high-dimensional, exponentially scaling Hilbert space. Furthermore, by manipulating experimental variables and measuring them in the low-dimensional space, one operates via a unitary operation on the states in the high-dimensional Hilbert space. This operation can be performed on a range of initial states (inputs), covering a region of the three phi-bit Hilbert space. This operation is predictable. Predictability results from the phi-bit response to parametric changes in the physical system. This three phi-bit unitary operation does not need to be decomposed in a sequence of smaller phi-bit gates. The output is measurable.

A logical phi-bit is a nonlinear acoustic mode supported by a metastructure composed of three elastically coupled finite length acoustic waveguides driven externally.[5] When the waveguides are subjected to two driving forces with two different frequencies, f_1 and f_2, a nonlinear mode "j" is well characterized by a frequency of the form $f^{(j)} = p^{(j)}f_1 + q^{(j)}f_2$, where $p^{(j)}$ and $q^{(j)}$ are integers. The corresponding acoustic displacement field at some location within the waveguide array (e.g., at one end of the guides) is fully defined by a 2×1 vector,

$$\vec{U}_{(j)} = \begin{pmatrix} \hat{c}_2 e^{i\varphi_{12}^{(j)}} \\ \hat{c}_3 e^{i\varphi_{13}^{(j)}} \end{pmatrix} e^{i\omega^{(j)}t}, \tag{1}$$

where the angular frequency $\omega^{(j)} = 2\pi f^{(j)}$. The magnitudes $\hat{C}_2$ and $\hat{C}_3$ are the magnitudes of the displacement of the second and third waveguides normalized to the magnitude of the first waveguide's displacement. $\varphi_{12}^{(j)} = \varphi_2^{(j)} - \varphi_1^{(j)}$ and $\varphi_{13}^{(j)} = \varphi_3^{(j)} - \varphi_1^{(j)}$ are the two independent phases in waveguides 2 and 3 relative to waveguide 1, respectively. The amplitudes and phases at the waveguide ends are controllable by tuning the driving frequencies and are also measurable unambiguously. The basis on which $\vec{U}_{(j)}$ is expressed, $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$, refers to the waveguides 2 and 3 of the array. We redefine the nontemporal part of the field in the subspace of the relative phases only by the normalized vector,

$$\vec{u}_{(j)} = \frac{1}{\sqrt{2}}\left(e^{i\varphi_{12}^{(j)}}\begin{pmatrix}1\\0\end{pmatrix} + e^{i\varphi_{13}^{(j)}}\begin{pmatrix}0\\1\end{pmatrix}\right), \tag{2}$$

where $\vec{u}_{(j)}$ is defined in a two-dimensional complex Hilbert space, $h_{(j)}$. Employing Dirac's ket notation to represent the basis vectors of that space, Eq. (2) reduces to

$$\vec{u}_{(j)} = \frac{1}{\sqrt{2}}\left(e^{i\varphi_{12}^{(j)}}|0\rangle_{(j)} + e^{i\varphi_{13}^{(j)}}|1\rangle_{(j)}\right). \tag{3}$$

A single phi-bit state, represented in this form, spans the Bloch sphere and is analogous to a quantum bit (qubit). Equation (3) is effectively a coherent superposition of $|0\rangle_{(j)}$ and $|1\rangle_{(j)}$ states with complex amplitudes $\frac{1}{\sqrt{2}}e^{i\varphi_{12}^{(j)}}$ and $\frac{1}{\sqrt{2}}e^{i\varphi_{13}^{(j)}}$.

Let us now consider three phi-bits. These three nonlinear modes, $j = 1, 2,$ and 3, result from the nonlinear mixing of the same two driving frequencies and are, therefore, strongly correlated. The six complex amplitudes associated with the three phi-bit individual states are dependent on the same driving frequencies. We can represent the state of a composite system of the three phi-bits via a tensor product of three single phi-bit states, namely,

$$\vec{V} = \vec{u}_{(1)} \otimes \vec{u}_{(2)} \otimes \vec{u}_{(3)}. \tag{4}$$

The state of this composite lives in the $2^3 = 8$ dimensional Hilbert space $H = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$, a tensor product space of individual phi-bit Hilbert spaces. $\vec{V}$ is defined on the basis $\{|0\rangle_{(1)}|0\rangle_{(2)}|0\rangle_{(3)}, |0\rangle_{(1)}|0\rangle_{(2)}|1\rangle_{(3)}, \ldots, |1\rangle_{(1)}|1\rangle_{(2)}|1\rangle_{(3)}\}$, commonly written as $\{|000\rangle, |001\rangle, \ldots, |111\rangle\}$. In writing these basis vectors, we have dropped the tensor product symbols for notation simplicity. The state vector $\vec{V}$ can, therefore, be defined as an 8×1 vector,

$$\vec{V} = \begin{pmatrix} e^{i\left(\varphi_{12}^{(1)} + \varphi_{12}^{(2)} + \varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{12}^{(1)} + \varphi_{12}^{(2)} + \varphi_{13}^{(3)}\right)} \\ e^{i\left(\varphi_{12}^{(1)} + \varphi_{13}^{(2)} + \varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)} + \varphi_{12}^{(2)} + \varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{12}^{(1)} + \varphi_{13}^{(2)} + \varphi_{13}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)} + \varphi_{12}^{(2)} + \varphi_{13}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)} + \varphi_{13}^{(2)} + \varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)} + \varphi_{13}^{(2)} + \varphi_{13}^{(3)}\right)} \end{pmatrix}. \tag{5}$$

We apply a transformation to the Hilbert space, H, which leads to a new basis $\{e^{i\left(\varphi_{12}^{(2)} + \varphi_{12}^{(3)}\right)}e^{-i\left(\frac{2}{3}\varphi_{12}^{(2)} + \frac{1}{3}\varphi_{12}^{(3)}\right)}|000\rangle, e^{i\left(\varphi_{12}^{(2)} + \varphi_{13}^{(3)}\right)} e^{-i\left(\frac{2}{3}\varphi_{12}^{(2)} + \frac{1}{3}\varphi_{13}^{(3)}\right)}|001\rangle, \ldots, e^{i\left(\varphi_{13}^{(2)} + \varphi_{13}^{(3)}\right)}e^{-i\left(\frac{2}{3}\varphi_{13}^{(2)} + \frac{1}{3}\varphi_{13}^{(3)}\right)}|111\rangle\}$. In this basis, the three phi-bit state vector takes the following form:

$$\vec{V}' = \begin{pmatrix} e^{i\left(\varphi_{12}^{(1)}+\frac{2}{3}\varphi_{12}^{(2)}+\frac{1}{3}\varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{12}^{(1)}+\frac{2}{3}\varphi_{12}^{(2)}+\frac{1}{3}\varphi_{13}^{(3)}\right)} \\ e^{i\left(\varphi_{12}^{(1)}+\frac{2}{3}\varphi_{13}^{(2)}+\frac{1}{3}\varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)}+\frac{2}{3}\varphi_{12}^{(2)}+\frac{1}{3}\varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{12}^{(1)}+\frac{2}{3}\varphi_{13}^{(2)}+\frac{1}{3}\varphi_{13}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)}+\frac{2}{3}\varphi_{12}^{(2)}+\frac{1}{3}\varphi_{13}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)}+\frac{2}{3}\varphi_{13}^{(2)}+\frac{1}{3}\varphi_{12}^{(3)}\right)} \\ e^{i\left(\varphi_{13}^{(1)}+\frac{2}{3}\varphi_{13}^{(2)}+\frac{1}{3}\varphi_{13}^{(3)}\right)} \end{pmatrix}. \quad (6)$$

Suppose that the three phi-bits have relative phases that vary in the same way upon tuning one of the driving frequencies. In that case, $\varphi_{12}^{(1)} = \varphi_{12}^{(2)} = \varphi_{12}^{(3)} = f(\Delta\nu)$ and $\varphi_{13}^{(1)} = \varphi_{13}^{(2)} = \varphi_{13}^{(3)} = g(\Delta\nu)$, where $\Delta\nu$ is a driving frequency tuning parameter. This parameter enables us to span some region of the Hilbert space, H, through the variations in the components of vector $\vec{V}'$. Let us further suppose that the functions f and g cross for some value of the tuning parameter $\Delta\nu^*$. At this tuning frequency, the state vector becomes

$$\vec{V}' = e^{i2f(\Delta\nu^*)}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

which to within a normalizing factor, and a general phase is the vector

$$\vec{V}'' = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

By applying the unitary transformation (i.e., applying a rotation to the Hilbert space)

$$\overleftrightarrow{U} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

to the vector $\vec{V}''$, one obtains a pure state vector in the space H,

$$\overleftrightarrow{U}\,\vec{V}'' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{V}^0. \quad (7)$$

Applying the unitary transformation, $\overleftrightarrow{U}$, to the general form of the state vector $\vec{V}'$ (i.e., for a tuning frequency different from $\Delta\nu^*$), one obtains

$$\vec{W} = \overleftrightarrow{U}\,\vec{V}' = \frac{1}{8}\begin{pmatrix} \left(e^{i\varphi_{12}^{(1)}} + e^{i\varphi_{13}^{(1)}}\right)z_2 \\ 2e^{i\left(\varphi_{12}^{(1)}+\frac{2}{3}\varphi_{12}^{(2)}\right)}z_0 \\ \sqrt{2}e^{i\varphi_{12}^{(1)}}z_3 \\ 2e^{i\left(\varphi_{12}^{(1)}+\frac{2}{3}\varphi_{13}^{(2)}\right)}z_0 \\ \left(e^{i\varphi_{13}^{(1)}} - e^{i\varphi_{12}^{(1)}}\right)z_2 \\ 2e^{i\left(\varphi_{13}^{(1)}+\frac{2}{3}\varphi_{12}^{(2)}\right)}z_0 \\ \sqrt{2}e^{i\varphi_{13}^{(1)}}z_3 \\ 2e^{i\left(\varphi_{13}^{(1)}+\frac{2}{3}\varphi_{13}^{(2)}\right)}z_0 \end{pmatrix}, \quad (8)$$

with

$$z_0 = e^{i\frac{1}{3}\varphi_{13}^{(3)}} - e^{i\frac{1}{3}\varphi_{12}^{(3)}},$$
$$z_1 = e^{i\frac{1}{3}\varphi_{12}^{(3)}} + e^{i\frac{1}{3}\varphi_{13}^{(3)}},$$
$$z_2 = \left(e^{i\frac{2}{3}\varphi_{12}^{(2)}} + e^{i\frac{2}{3}\varphi_{13}^{(2)}}\right)z_1,$$
$$z_3 = \left(e^{i\frac{2}{3}\varphi_{13}^{(2)}} - e^{i\frac{2}{3}\varphi_{12}^{(2)}}\right)z_1.$$

$\vec{W}$ in Eq. (8) is a new representation of the three phi-bit state in H. This vector is nonseparable (i.e., classically entangled) for most values of the phase differences. The 8×1 vector cannot be factored into the tensor product of three 2×1 vectors. Note that here we do not need entanglement like in quantum mechanics to achieve correlations between subsystems in multipartite composite systems. The phi-bits are naturally nonlinearly correlated. Nonseparability is used to enable a more complete coverage of the multi phi-bit Hilbert space. The procedure described above related the six experimentally controllable and measurable variables $\varphi_{12}^{(1)}$, $\varphi_{13}^{(1)}$, $\varphi_{12}^{(2)}$, $\varphi_{13}^{(2)}$, $\varphi_{12}^{(3)}$, and $\varphi_{13}^{(3)}$ to the eight components of Eq. (8). In other words, it relates a space of phase differences that scales linearly with the N logical phi-bits to an exponentially scaling complex space of dimension 2^N.

By tuning the driving frequencies, i.e., manipulating the phase difference in the linear space, one acts or effectively operates on vectors in the exponentially complex space. Furthermore, recall that since we have chosen phi-bits such that the six phase differences relate to only two tunable functions f and g, we are relating a two-dimensional physical space to a 2^3-dimensional space. The two-dimensional space can be spanned with a single tuning frequency parameter. This single parameter operates by rotating the 8-dimensional complex state vector of the three phi-bits. Let us illustrate this type of operation.

Expressing $\vec{W}$ in terms of f and g yields

$$\vec{W} = \frac{1}{8}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \end{pmatrix}, \tag{9}$$

with

$$\begin{aligned}
s_1 &= \left(e^{if} + e^{ig}\right)\left(e^{i\frac{1}{3}f} + e^{i\frac{1}{3}g}\right)\left(e^{i\frac{2}{3}f} + e^{i\frac{2}{3}g}\right),\\
s_2 &= 2e^{if}\left(e^{i\frac{1}{3}g} - e^{i\frac{1}{3}f}\right),\\
s_3 &= \sqrt{2}e^{if}\left(e^{i\frac{2}{3}g} - e^{i\frac{2}{3}f}\right)\left(e^{i\frac{1}{3}f} + e^{i\frac{1}{3}g}\right),\\
s_4 &= 2e^{i\left(f+\frac{2}{3}g\right)}\left(e^{i\frac{1}{3}g} - e^{i\frac{1}{3}f}\right),\\
s_5 &= \left(e^{ig} - e^{if}\right)\left(e^{i\frac{1}{3}f} + e^{i\frac{1}{3}g}\right)\left(e^{i\frac{2}{3}f} + e^{i\frac{2}{3}g}\right),\\
s_6 &= 2e^{i\left(g+\frac{2}{3}f\right)}\left(e^{i\frac{1}{3}g} - e^{i\frac{1}{3}f}\right),\\
s_7 &= \sqrt{2}e^{ig}\left(e^{i\frac{2}{3}g} - e^{i\frac{2}{3}f}\right)\left(e^{i\frac{1}{3}f} + e^{i\frac{1}{3}g}\right),\\
s_8 &= 2e^{ig}\left(e^{i\frac{1}{3}g} - e^{i\frac{1}{3}f}\right).
\end{aligned}$$

Let us now assume that the functions f and g take the form illustrated in Fig. 1. Although the figure shows linear functions, the functions f and g do not need to be linear but only need to cross and be symmetric about the crossing point.

Upon tuning one of the driving frequencies from $\Delta\nu_1$ and $\Delta\nu_2$, one swaps the values of f and g in the low-dimension space of phase difference functions. This physical operation results in a change of the 8-dimensional state vector,

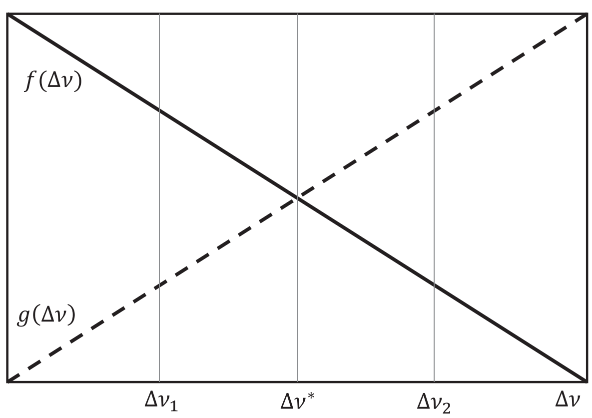

FIG. 1. Schematic representation of phase difference functions as the frequency tuning parameter $\Delta\nu$. At $\Delta\nu^*$, one has $f = g$. $\Delta\nu_1$ and $\Delta\nu_2$ are tuning parameters corresponding to a swap of the values of f and g.

$$\vec{W}' = \frac{1}{8}\begin{pmatrix} s_1 \\ -s_8 \\ -s_7 \\ -s_6 \\ -s_5 \\ -s_4 \\ -s_3 \\ -s_2 \end{pmatrix}, \tag{10}$$

where $\vec{W}'$ is related to $\vec{W}$ by a unitary transformation, $\overleftrightarrow{T}$, such that $\vec{W}' = \overleftrightarrow{T}\,\vec{W}$ with

$$\overleftrightarrow{T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

This nontrivial three phi-bit operation in the exponentially complex 8-dimensional Hilbert space reorders the last seven components of the state vector and adds a π phase to the permuted components. Note that this operation is independent of the specific value of f and g. One can choose different values for $\Delta\nu_1$ corresponding to a range of possible 8×1 input vectors and apply the transformation $\overleftrightarrow{T}$ by tuning the frequency parameter to $\Delta\nu_2 = \Delta\nu^* + \Delta\nu_1$ to obtain the corresponding 8×1 output vector. This operation can be achieved by simply monitoring the value of the phase differences f and g. Since phi-bits are classical entities, one can operate on them while continuously measuring their phase characteristics. The $\overleftrightarrow{T}$ operation is completely input independent as long as the input belongs to the accessible region of the Hilbert space, H.

To contrast phi-bit-based computing with quantum computing,[1,2] we show in Fig. 2 a quantum circuit that operates in the same way as the three phi-bit unitary matrix, $\overleftrightarrow{T}$. A quantum circuit is a sequence of quantum gates that operates on qubits to enable a quantum computation. This circuit operates on three qubit states. Simultaneous operations are tensor products of single qubit gates, such as the Pauli X, Z, and Hadamard (H) gates. Also, this circuit involves three qubit Toffoli gates that are often only realizable as circuits of single and two qubit operations.[7] The first block of the circuit applies the π phase change to the appropriate components of the 8×1 input vector. The second circuit block swaps components of the vector. Note that the circuit in Fig. 2 is not unique, and there are multiple such circuits that produce the operation of $\overleftrightarrow{T}$.

The three phi-bit unitary transformation (i.e., three phi-bit gate), $\overleftrightarrow{T}$, does not need to be decomposed into a sequence of smaller gates to be physically realized. It is an example of a full muti phi-bit operation that can be conducted in a single manipulation of the experimental system.

The experimental setup consists of a metastructure composed of elastically coupled aluminum rod-like acoustic waveguides.[5] The rods

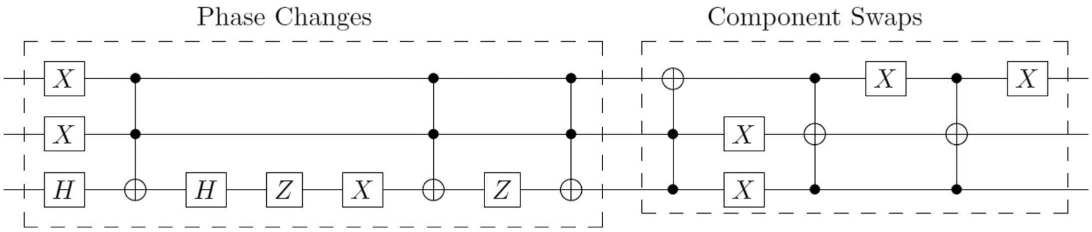

FIG. 2. Example of a three qubit quantum circuit that leads to the same unitary operation as the transformation $\overleftrightarrow{T}$. The horizontal lines correspond to qubits. X, Z, and H are the one qubit Pauli X, Pauli Z, and Hadamard gates. This circuit also employs six Toffoli gates. Note that these 3-qubit gates are often decomposed into smaller gates. Absence of gates on a qubit line corresponds to the identity operation.

are arranged in a linear array coupled along their length with epoxy resin. Three separate signal generators and amplifiers are used to drive piezoelectric transducers. Driving and detecting transducers are attached to the opposite ends of the rods with ultrasonic coupling agent. The signals generated by the detecting transducers enter an oscilloscope via independent input channels for the measurement of phases. The array of waveguides is suspended by thin threads for isolation. When driven externally at two primary frequencies, this metastructure supports a displacement field, which, when partitioned into the frequency domain, leads to modes with secondary frequencies associated with logical phi-bits. Rod 1 is driven at a frequency $f_1 = 62\,\text{kHz} + \Delta\nu$, and rod 2 at a frequency $f_2 = 70\,\text{kHz}$. $\Delta\nu$ is varied by increments of 50 Hz in the interval $[0, 8\,\text{kHz}]$. We focus our attention on three phi-bit modes, $j = 1, 2, \text{and}\, 3$, with frequencies $f^{(1)} = 4f_1 - 3f_2$, $f^{(2)} = 4f_1 - 2f_2$, and $f^{(3)} = 4f_1 - f_2$, respectively. These three phi-bits are chosen to have the same coefficients, $p^{(j)}$. The six measured phases, $\varphi_{12}^{(j)}$ and $\varphi_{13}^{(j)}$, exhibit several remarkable features in the form of upward or downward jumps superposed onto monotonous background variations. The jumps amount to phase changes on the order of π (180°). In Fig. 3, the measured phases $\varphi_{12(3)}^{(j)}$ of phi-bits $j = 1$, 2, and 3 have been corrected by a translation of $+q^{(j)} \times C_{12(3)}$, where the $C_{12(3)}$ are constants to overlap the background variations.

The origin of the phase jumps and background variations in terms of nonlinear interactions of driven acoustic waves in the metastructure was discussed in detail previously.[8,9] For all logical phi-bits, the background phases show variations as a function of frequency of several thousand Hz. This background was shown theoretically in Refs. 8 and 9 to possibly originate from extrinsic nonlinear effects associated with the amplifiers and/or transducers and/or coupling agent that enable the driving and characterization of the physical system. We proposed a model of the array of waveguides coupled to extrinsic nonlinear damped oscillators at the rod ends that may be representative of the nonlinearity of the electronics/transducer/ultrasonic-couplant assembly. The order of nonlinearity was chosen to be an integer $Q = p + q$, a sum of two other integers, p and q. Since the nonlinear oscillators are physically in contact with the ends of the rods, we showed that they contribute to the detected signals at the ends of the waveguides in the form of a background to the phases of a logical phi-bit "i" with frequency $f^{(j)} = p^{(j)}f_1 + q^{(j)}f_2$ that can be expressed as a linear combinations $p\varphi_{12}(f_1) + q\varphi_{12}(f_2)$ and $p\varphi_{13}(f_1) + q\varphi_{13}(f_2)$ of the phase differences associated with the linear displacement of the array of waveguides at the frequencies f_1 and f_2. These

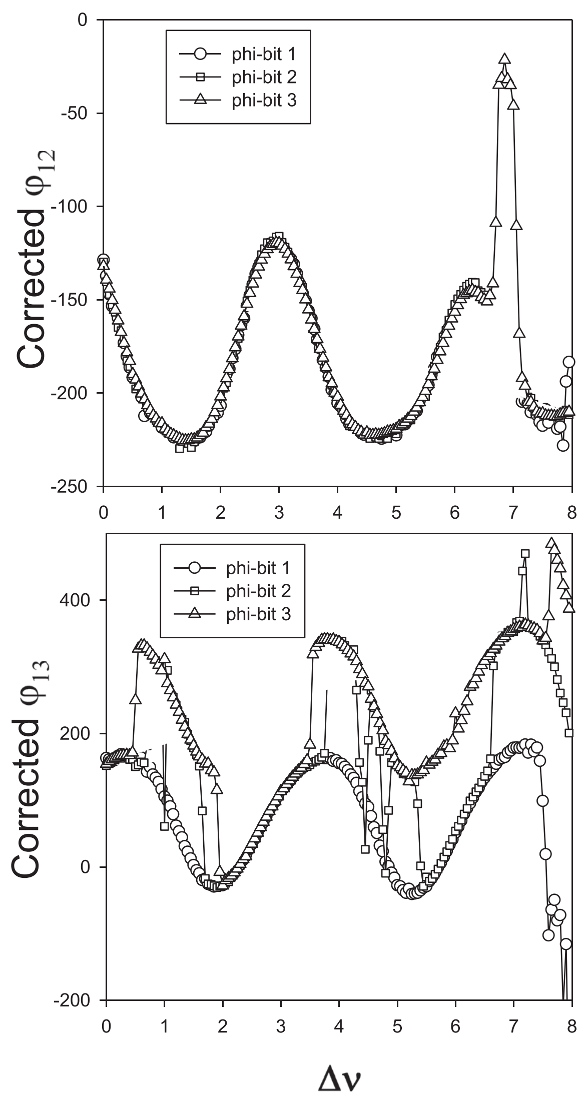

FIG. 3. Corrected measured phases of three phi-bits as functions of the driving frequency tuning parameter $\Delta\nu$. The phases exhibit phase jumps overlapping with common background monotonous variations.

variations will, therefore, be controlled by the characteristic frequencies of the linear modes of vibration of the array of waveguides. Since these characteristic frequencies are known experimentally to be spaced by several kilohertz,[3] the background phase differences exhibit variations with the same frequency scale as that of the linear modes as shown in Fig. 3.

We also proposed a model of the array of waveguides with intrinsic cubic nonlinearity and damping arising from the epoxy resin coupling the rods along their length. Multiple timescale perturbation theory was used to show that the amplitude-frequency response of a super-harmonic nonlinear resonant mode $\{p = 2, q = -1\}$ behaved like that of a Duffing nonlinear oscillator.[10] The amplitude-response presented an overhang that can lead to a sharp phase jump that approaches π. This type of jump exhibits hysteretic behavior, which was also observed experimentally.[8] Similar behavior may occur for orders of nonlinearity going beyond cubic nonlinearity.

By subtracting easily identifiable contributions of the phase jumps to the overall experimental data, we can extract the background contribution to the phases common to the three phi-bits. Figure 4 reports the common background variations $\varphi_{12}^{\prime(1)} = \varphi_{12}^{\prime(2)} = \varphi_{12}^{\prime(3)} = f(\Delta\nu)$ and $\varphi_{13}^{\prime(1)} = \varphi_{13}^{\prime(2)} = \varphi_{13}^{\prime(3)} = g(\Delta\nu)$. Since phases are always relative, in Fig. 4, the background phase, g, has been obtained by subtracting a general phase of 71° to achieve overlap with f. Note that these corrections, subtractions, and translations are well-defined and controllable algebraic manipulations in the low-dimensional physical space of the system. They do not affect the generality of the mapping between the three phi-bit physical space and the associated exponentially complex Hilbert space.

Figure 4 shows that we can experimentally achieve conditions similar to those described in Fig. 1 in the vicinity of the tuning frequency $\Delta\nu^* \sim 3.33kHz$. Note that here one does not need the functions f and g to be linear as was illustrated in the theoretical case of Fig. 1. The approximate symmetry of the experimentally derived functions f and g about $\Delta\nu^*$ in the range of tuning frequencies $\Delta\nu \in [2.8,\ 3.9\text{kHz}]$ enables us to use the three phi-bits, 1, 2, and 3, to realize experimentally the theoretically predicted unitary operation $\overleftrightarrow{T}$. Note also that the range of $\Delta\nu$ to the left of $\Delta\nu^*$ of approximately 3 kHz produces different values of f and g, i.e., initial states, on which one can operate. These different values of f and g inserted into Eq. (9) correspond to the accessible region of the Hilbert space, H, i.e., the state vectors, $\vec{W}$, on which the transformation $\overleftrightarrow{T}$ can operate. This operation is completely independent of the specific value of the input within this accessible region of H. By symmetry, outputs $\vec{W}'$ will span another region of the Hilbert space corresponding to the exchanged values of f and g on the right side of $\Delta\nu^*$.

These experimental data demonstrate the existence of logical phi-bits supported by an externally driven acoustic waveguide metastructure that can be used to realize a nontrivial three phi-bit unitary operation that would be challenging to realize in a quantum computer.

This result establishes a correspondence between the state of correlated logical phi-bits represented in a low-dimensional linearly scaling physical space and their state representation in a high-dimensional, exponentially scaling Hilbert space. The logical phi-bits are nonlinear acoustic waves, classical analogues of qubits, which can be supported by a metastructure constituted of an externally driven array of acoustic waveguides. By manipulating the physical variables and measuring them in the physical space of three phi-bits with at most $2 \times 3 = 6$ dimensions, one can operate on the coherent superposition of states in the $2^3 = 8$ dimensional Hilbert space via a three phi-bit nontrivial unitary operation, i.e., a three phi-bit gate. This operation can be performed on a range of input states covering a nontrivial region of the three phi-bit Hilbert spaces. This three phi-bit gate can be represented by a 8×8 unitary matrix that swaps seven components of the three phi-bits 8×1 state vector and adds a π phase to the permuted components. Realizing this unitary operation in a quantum computer would involve the challenging task of applying a long sequence of smaller single and two qubit gates to a three qubit system. We have also shown that the conditions to physically realize the three phi-bit gate exist in an experimental laboratory scale metastructure composed of coupled rod-like acoustic waveguides. In contrast to multipartite quantum systems, the coherent superpositions of states of the

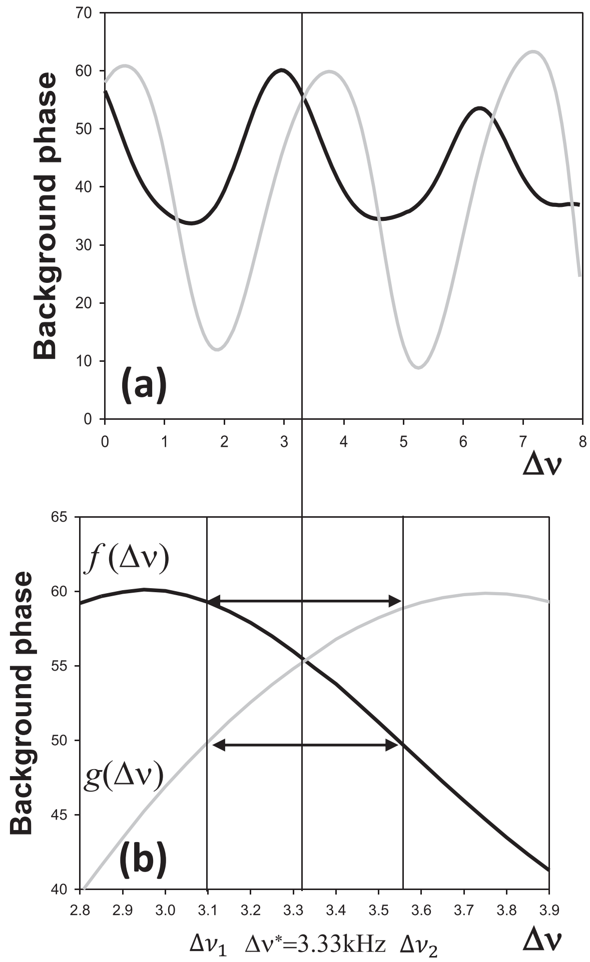

FIG. 4. (a) Experimental common background phases of the three phi-bits corresponding to $\varphi_{12}^{\prime(1)}$, $\varphi_{12}^{\prime(2)}$, and $\varphi_{12}^{\prime(3)}$ (black solid line) and $\varphi_{13}^{\prime(1)}$, $\varphi_{13}^{\prime(2)}$, and $\varphi_{13}^{\prime(3)}$ (gray solid line) extracted from the data of Fig. 3. (b) Magnification of the experimental data near $\Delta\nu^* \sim 3.33kHz$. The black and gray solid lines correspond to the functions $f(\Delta\upsilon)$ and $g(\Delta\upsilon)$, respectively. The thin black vertical line corresponds to $\Delta\nu^* \sim 3.3kHz$, where $f = g$. The thick gray vertical lines show how tuning frequency from $\Delta\nu_1$ to $\Delta\nu_2$ can experimentally swap the values of f and g (double arrows) in a way similar to the theoretical behavior of Fig. 1.

three logical phi-bits system on which one can operate are stable, do not decohere, and are directly measurable.

This work serves as proof-of-concept that one can realize, with a nonlinear acoustic metastructure, quantum-like coherent superpositions of states in an exponentially complex Hilbert space on which one can effectively operate with a nontrivial unitary operation analogous to a quantum gate. Future work will consist of extending the domain of input superpositions of states in the 8-dimensional Hilbert space on which the transformation $\overleftrightarrow{T}$ can operate. Additional work will involve investigating the scalability of the approach to more than three phi-bits and operate on any number $N \geq 3$ of logical phi-bits in exponentially scaling complex spaces of dimension, 2^N. Finally, we are evaluating other purposeful approaches in the design of multi-logical phi-bit gates with the aim of using them in the development of efficient algorithms.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Pierre A. Deymier: Conceptualization (equal); Writing – original draft (equal). **Keith Runge:** Conceptualization (equal); Writing – review & editing (equal). **Philippe Cutillas:** Formal analysis (equal); Writing – review & editing (equal). **M. Arif Hasan:** Data curation (equal); Writing – review & editing (equal). **Trevor D. Lata:** Formal analysis (equal); Resources (equal); Writing – review & editing (equal). **Joshua A. Levine:** Visualization (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

REFERENCES

[1] J. D. Hidary, *Quantum Computing: An Applied Approach* (Springer, 2019).
[2] I. B. Djordjevic, *Quantum Information Processing, Computing and Error Correction*, 2nd ed. (Academic Press, 2021).
[3] M. A. Hasan, L. Calderin, T. Lata, P. Lucas, K. Runge, and P. A. Deymier, "The sound of bell states," Commun. Phys. **2**, 106 (2019).
[4] L. Calderin, M. Arif Hasan, N. Jenkins, T. Lata, P. Lucas, K. Runge, and P. A. Deymier, "Experimental demonstration of coherent superpositions in an ultrasonic pseudospin," Sci. Rep. **9**, 14156 (2019).
[5] M. A. Hasan, K. Runge, and P. A. Deymier, "Experimental classical entanglement in a 16 acoustic qubit-analogue," Sci. Rep. **11**, 24248 (2021).
[6] K. Runge, M. A. Hasan, J. A. Levine, and P. A. Deymier, "Demonstration of a two-bit controlled-NOT quantum-like gate using classical acoustic qubit-analogues," Sci. Rep. **12**, 14066 (2022).
[7] G. F. Viamontes, I. L. Markov, and J. P. Hayes, *Quantum Circuit Simulation* (Springer, Dordrecht, 2009).
[8] P. A. Deymier, K. Runge, M. A. Hasan, T. D. Lata, J. A. Levine, and P. Cutillas, "Realizing acoustic qubit analogues with nonlinearly tunable phi-bits in externally driven coupled acoustic waveguides," Sci. Rep. **13**, 635 (2023).
[9] P. A. Deymier, K. Runge, M. A. Hasan, T. D. Lata, and J. A. Levine, "Tuning logical phi-bit state vectors in an externally driven non-linear array of acoustic waveguides via drivers' phase," Quantum Rep. (unpublished).
[10] S. Yang, A. H. Nayfeh, and D. T. Mook, "Combination resonances in the response of the Duffing oscillator to a three-frequency excitation," Acta Mech. **131**, 235 (1998).