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Non-separable states in a bipartite elastic system

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We consider two one-dimensional harmonic chains coupled along their length via linear springs. Casting the elastic wave equation for this system in a Dirac-like form reveals a directional representation. The elastic band structure, in a spectral representation, is constituted of two branches corresponding to symmetric and antisymmetric modes. In the directional representation, the antisymmetric states of the elastic waves possess a plane wave orbital part and a 4x1 spinor part. Two of the components of the spinor part of the wave function relate to the amplitude of the forward component of waves propagating in both chains. The other two components relate to the amplitude of the backward component of waves. The 4x1 spinorial state of the two coupled chains is supported by the tensor product Hilbert space of two identical subsystems composed of a non-interacting chain with linear springs coupled to a rigid substrate. The 4x1 spinor of the coupled system is shown to be in general not separable into the tensor product of the two 2x1 spinors of the uncoupled subsystems in the directional representation. © *2017 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).* [http://dx.doi.org/10.1063/1.4982732]

I. INTRODUCTION

The phenomenon of entanglement of quantum systems is receiving an increasing amount of attention in the light of its importance in the development of quantum information science and computing.[1] Entangled quantum states are those states, defined in a vector space obtained by a tensor product of subspaces, which cannot be written as a tensor product of states in the subspaces. One can distinguish between two forms of entanglement, namely entanglement between spatially separated systems or particles and entanglement between different degrees of freedom of the same system or particle.[2,3] The former type of entanglement is nonlocal and exclusively quantum in nature, while the latter is local and may occur in classical systems. For example, bipartite classical entanglement has been achieved between the orbital angular momentum and the polarization states of electromagnetic waves.[4,5] Tripartite classical entanglement has also been reported by controlling the path, polarization, and transverse mode degrees of freedom of a classical laser beam.[6] From the point of view of nomenclature, since classical entanglement does not account for non-locality, it is more appropriate to use the term "non-separability" and describe correlated states of a single systems as "non-separable states".[7]

While non-separability in the states of electromagnetic waves has been extensively studied, the objective of the present paper is to address the much less studied notions of separability and non-separability of multipartite classical mechanical systems supporting elastic waves. More specifically, we consider separability relative to the choice of the partitioning of the multipartite system. Indeed, it is known that given a multipartite physical system, whether quantum or classical, the way to subdivide it into subsystems is not unique.[8] Therefore, the separability and non-separability of multipartite quantum systems possess some ambiguity relative to their decomposition into subsystems. For instance, the states of a quantum system may not appear entangled relative to some decomposition but appear entangled relative to another partitioning. The question arises as to how to make the choice of a decomposition into subsystems of a classical elastic systems. The criterion for that choice may be the ability to perform observations and measurements of some degrees of freedom of the subsystems.[9]

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In particular, we consider a bipartite classical mechanical system composed of two coupled one-dimensional elastic chains whose elastic wave equations can be factored into a Dirac-like equation. The antisymmetric elastic waves exhibit quantum-like behavior. Indeed, the elastic wave functions, solutions of the Dirac-like equation, possess a plane wave orbital part and an amplitude that takes on the form of spinors. The 4x1 spinor amplitude expressed in a directional representation leads to constraints on the degrees of freedom associated with the direction of propagation along the chains. The components of the spinor part of the wave function are also measurable. One can establish a one-to-one correspondence between a measurable quantity, namely the transmission coefficient along the chains and the components of the spinor part of the wave function.[10–12] While the states of the bipartite mechanical system are separable in a spectral representation, the possibility of measuring the spinor part of its wave function dictates another partitioning into two identical subsystems, each composed of a single elastic chain coupled to a rigid substrate. The states of elastic waves in these subsystems can also be described via a Dirac-like equation and possess 2x1 spinor amplitudes. These amplitudes are also again measurable through measurements of transmission coefficients. The 4x1 spinor states of the elastic bipartite system, defined in the tensor product Hilbert space of the two subsystems, is shown not to be expressible as tensor products of subsystems 2x1 spinor states except in some limiting cases. The non-separability of the states in terms of direction of propagation of the elastic bipartite system relative to single chain subsystems is analogous to the phenomenon of local correlation. These non-separable classical states and their analogy with local quantum states may prove to be useful in quantum information processing.

II. MODEL SYSTEMS

A. Coupled two-chain system

We consider two one-dimensional harmonic chains with linear springs coupling pairs of masses along the chains (see figure 1a).

In the long wavelength limit, the system is treated as continuous and the equations of motion are:

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2 \frac{\partial^2}{\partial x^2}\right) u + \gamma^2 (u - v) = 0 \tag{1}$$

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2 \frac{\partial^2}{\partial x^2}\right) v - \gamma^2 (u - v) = 0 \tag{2}$$

u and v are the displacements in the top chain "a" and the bottom chain "b" respectively. The constants β and γ depend on the stiffness of the spring and the masses. Inserting plane wave solutions:

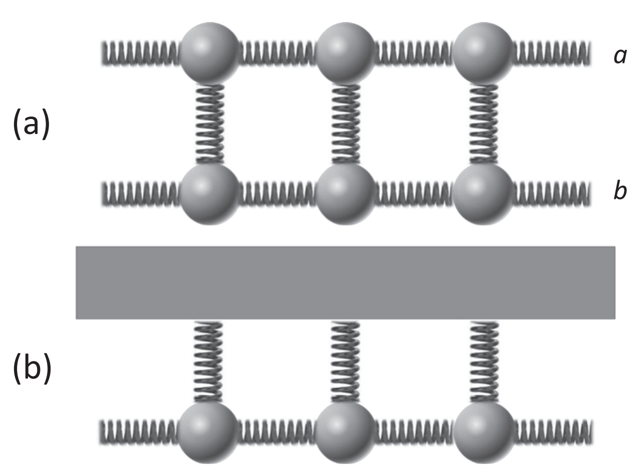

FIG. 1. Schematic illustration of (a) the two-chain elastic system. The top and bottom chains are identified as "a" and "b", respectively; and (b) system composed of a single chain coupled to a rigid substrate via side springs.

$u = Ae^{i\omega t}e^{ikx}$ and $v = Be^{i\omega t}e^{ikx}$ yields the eigen value problem:

$$\begin{cases}\left(\omega^2 - \beta^2k^2 - \gamma^2\right)A + \gamma^2 B = 0\\ \gamma^2 A + \left(\omega^2 - \beta^2k^2 - \gamma^2\right)B = 0\end{cases} \tag{3}$$

There are two sets of eigen values:

$$\omega^2 = \beta^2 k^2 \tag{4}$$

and

$$\omega^2 = \beta^2 k^2 + 2\gamma^2 \tag{5}$$

These eigen values correspond to two nondegenerate dispersion curves in the one-dimensional band structure of the coupled two-chains system. The first eigen value (Eq. (4)) passes through the origin and is the same as that of the uncoupled chains. Inserting its expression into equation (3) leads to the condition on the amplitudes: $A = B$. This is the well know symmetric mode where the displacements $u(x, t)$ and $v(x, t)$ of the masses in the two chains are in phase. There is no energy stored in the coupling springs. Using the second eigen value which takes on a finite value at the origin, leads to the condition on the amplitudes: $A = -B$. This is the antisymmetric mode. The displacements of the masses in the two chains are out of phase (phase difference of π). There is energy exchanged and stored in the coupling springs.

To analyze this system further, let us consider the problem of the coupled chains using Dirac-like forms of the equations of motion. In the long wavelength limit, the Dirac-like equation for this coupled system is written in the form:

$$\left[\boldsymbol{\sigma_x} \otimes \boldsymbol{\sigma_x}\frac{\partial}{\partial t} + i\beta\boldsymbol{\sigma_x} \otimes \boldsymbol{\sigma_y}\frac{\partial}{\partial x} - i\delta\boldsymbol{C}\right]\Psi = 0 \tag{6a}$$

$$\left[\boldsymbol{\sigma_x} \otimes \boldsymbol{\sigma_x}\frac{\partial}{\partial t} + i\beta\boldsymbol{\sigma_x} \otimes \boldsymbol{\sigma_y}\frac{\partial}{\partial x} + i\delta\boldsymbol{C}\right]\bar{\Psi} = 0 \tag{6b}$$

with $\boldsymbol{C} = \begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix} \otimes \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$ and $\delta = \frac{1}{\sqrt{2}}\gamma$. $\boldsymbol{\sigma_x}$ and $\boldsymbol{\sigma_y}$ are the 2x2 Pauli matrices given by $\boldsymbol{\sigma_x} = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$ and $\boldsymbol{\sigma_y} = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}$. In the language of quantum field theory the non-self dual Ψ and $\bar{\Psi}$ are 4x1 vectors, corresponding to "particles" and "antiparticles." In matrix form and for the minus part of the $\pm$ ("particles"), equation (6a) becomes:

$$\left[\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\frac{\partial}{\partial t} + \beta\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\frac{\partial}{\partial x} - i\delta\begin{pmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{pmatrix}\right]\Psi = 0 \tag{7}$$

Introducing a k-dependent function, $\varphi(k, t)$, $\Psi = \varphi(k, t)e^{ikx}$, we can rewrite equation (7) in a spectral representation:

$$\left(\boldsymbol{\Sigma}_x\frac{\partial}{\partial t} + i\boldsymbol{U}_k\right)\varphi = 0 \tag{8}$$

where $\boldsymbol{\Sigma}_x = \boldsymbol{\sigma_x} \otimes \boldsymbol{\sigma_x}$ and $\boldsymbol{U}_k = i\beta k\boldsymbol{\sigma_x} \otimes \boldsymbol{\sigma_y} - \delta\boldsymbol{C}$.

B. Single chain coupled to a substrate

We also consider the case of a single chain coupled to a rigid substrate through harmonic springs (Fig. 1b) to illustrate our reference for the decomposition of the coupled two-chain system. The long-wavelength equation of motion for the single chain system is:

$$\frac{\partial^2 u}{\partial t^2} - \beta^2\frac{\partial^2 u}{\partial x^2} + \alpha^2 u = 0 \tag{9}$$

Equation (9) can be also factored in the form of Dirac equations and its adjoint:

$$\left[\boldsymbol{\sigma_x}\frac{\partial}{\partial t} + i\beta\boldsymbol{\sigma_y}\frac{\partial}{\partial x} - i\alpha\boldsymbol{I}\right]\psi = 0 \tag{10a}$$

$$\left[\boldsymbol{\sigma_x}\frac{\partial}{\partial t} + i\beta\boldsymbol{\sigma_y}\frac{\partial}{\partial x} + i\alpha\boldsymbol{I}\right]\bar{\psi} = 0 \tag{10b}$$

where $\boldsymbol{\sigma_x}$ and $\boldsymbol{\sigma_y}$ are again the 2x2 Pauli matrices and $\boldsymbol{I}$ is the 2x2 identity matrix. We have investigated that system in a previous publication.[11] Here, we have written the solutions to Dirac equation in the form:

$$\psi_k = \psi(k,\omega_k) = c_0\xi_k\,(k,\omega_k)\,e^{(\pm)i\omega_k t}e^{(\pm)ikx}$$

where ξ_k are two by one spinors. It is important to point out that the interpretation of the Dirac spinor is in terms of forward and backward propagating waves.

Two limits of the spinor components of the single chain system will be useful in section III.B which are listed in Table II.

We have shown in reference 10–12, that the spinor part of the wave function when projected on the orthonormal basis $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ represents a superposition of states in the possible directions of propagation of the wave. In such a directional representation, these states are quasi-standing waves. If one solves the classical Eq. (9) using plane wave solutions, one does find that the directions of propagation are correlated as required for quasi-standing waves. The amplitude of the forward propagating and backward propagating waves that make up the quasi-standing wave are not independent of each other. Introducing the Dirac formalism enables us to project the quasi-standing wave solutions on forward and backward propagating states.

We also note that in conventional quantum systems, a measurement on a superposition of states would collapse the wave function into a pure state. A number of measurements is then required to obtain probabilistic information on the characteristics of the superposition of states. We have shown[11] that one can use measurement of the transmission coefficient to determine the spinor part of the wave function in the directional representation and through a single measurement determine the system's superposition of states. Indeed, by defining the number operator: $N = \int dx\psi^{\dagger}\boldsymbol{\sigma}_x\psi$ where ψ is promoted to an operator and $\psi^{\dagger}$ is its Hermitian conjugate. These operators when expanded in plane waves can be expressed in terms of creation and annihilation operators, namely $a_k^{\dagger}, a_k^{\dagger *}, a_k$, and a_k^{*}, such that $\psi\,(x) = \sum_k \frac{1}{\sqrt{2\omega}}\left[a_k\xi_k e^{ikx}e^{-i\omega t} + a_k^{*}\xi_k^{*}e^{-ikx}e^{i\omega t}\right]$ and $\psi^{\dagger}(x) = \sum_k \frac{1}{\sqrt{2\omega}}\left[a_k^{\dagger}\xi_k^{\dagger}e^{-ikx}e^{i\omega t} + a_k^{\dagger *}\xi_k^{\dagger *}e^{ikx}e^{-i\omega t}\right]$. We showed that $\boldsymbol{N} = \sum_k \frac{1}{2\omega}\left(a_k^{\dagger}a_k\xi_k^{\dagger}\boldsymbol{\sigma_x}\xi_k + b_k^{\dagger}b_k\eta_k^{\dagger}\boldsymbol{\sigma_x}\eta_k\right) = \sum_k\left(a_k^{\dagger}a_k + b_k^{\dagger}b_k\right)$ is an invariant. We can rewrite the number operator for wave number k, as: $N_k = \frac{1}{2\omega}a_k^{\dagger}a_k\xi_k^{\dagger}\boldsymbol{I}\boldsymbol{\sigma_x}\xi_k = a_k^{\dagger}a_k\xi_k^{\dagger}(\frac{1}{2\omega}S_+S_-\sigma_x + \frac{1}{2\omega}S_-S_+\sigma_x)\xi_k$ where we use the direction switching operators $\boldsymbol{S}_+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ and $\boldsymbol{S}_- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$. Separating the orbital part of the wave function with creation and annihilation operators from the spinorial part defines the operator $\frac{1}{2\omega}S_+S_-\sigma_x$. This operator represents the occupancy of one of the directions of propagation along the chain of mass and springs. Its Eigen values are given by: $n_k^+ = \xi_k^{\dagger}\frac{1}{2\omega}S_+S_-\sigma_x\xi_k = \frac{\omega+\beta k}{2\omega}$. The operator $\frac{1}{2\omega}S_-S_+\sigma_x$ corresponds to the occupancy of the opposite direction of propagation. Its Eigen values are given by: $n_k^- = \xi_k^{\dagger}\frac{1}{2\omega}S_-S_+\sigma_x\xi_k = \frac{\omega-\beta k}{2\omega}$. The transmission coefficient is a measure of these eigen values through the relation (n_k^+ - n_k^-). For k=0, we find $n_k^+ = n_k^- = \frac{1}{2}$ which represents a standing wave with zero transmission (n_k^+ - n_k^-)=0. For k=∞, we find $n_k^+ = 1$, $n_k^- = 0$ which represents a traveling wave wave with complete transmission (n_k^+ - n_k^-)=1.

III. SEPARABILITY AND NON-SEPARABILITY OF MULTIPARTITE COMPOSITE SYSTEMS

A. The two-chain system is separable in the spectral representation

Let us now consider a bipartite composite system constituted of two two-chain systems in two different non-degenerate states, $\varphi^1 = \varphi(k_1, t)$ and $\varphi^2 = \varphi(k_2, t)$. We rewrite equation (8) in the form of the equation for a two non-interacting two-chain system:

$$\left(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_x \frac{\partial}{\partial t} + i\boldsymbol{\Sigma}_x \otimes \boldsymbol{U}_k + i\boldsymbol{U}_k \otimes \boldsymbol{\Sigma}_x\right)\phi = 0 \tag{11}$$

The solution $\phi = \varphi^1 \otimes \varphi^2$ is now a 16x1 vector representing the tensor product of the states of the two two-chain systems. This can be generalized to a multipartite composite system of N non-interacting two-chain systems in the various states: $\varphi^1(k_1, t), \varphi^2(k_2, t), \ldots, \varphi^N(k_N, t)$. Equation (11) is generalized to:

$$\left(\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_x \ldots \boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_x \frac{\partial}{\partial t} + i\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_x \ldots \boldsymbol{\Sigma}_x \otimes \boldsymbol{U}_k + i\boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_x \ldots \boldsymbol{U}_k \otimes \boldsymbol{\Sigma}_x \right.$$
$$\left. + \ldots i\boldsymbol{U}_k \otimes \boldsymbol{\Sigma}_x \otimes \ldots \boldsymbol{\Sigma}_x \otimes \boldsymbol{\Sigma}_x\right)\Phi = 0 \tag{12}$$

The states of the N two-chain systems span the N tensor product space $H \otimes H \otimes \ldots \otimes H$ where H is the Hilbert space of the states of a single two-chain system in the spectral representation. It is straightforward to show that these states take the form of tensor products of single two-chain system states: $\Phi = \varphi^1 \otimes \varphi^2 \otimes \varphi^3 \otimes \ldots \otimes \varphi^N$. The multipartite system of N non-interacting two-chain systems is separable in the spectral representation.

This argument could also be applied to multiple single chain systems to show that a composite system of N non-interacting single chain systems is also separable in the spectral representation.

B. The two-chain system is not always separable in the directional representation

We solve equation (7) by choosing plane wave solutions, $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$ with $\Psi_j = a_j e^{+i\omega t} e^{+ikx}$. The functions $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \times e^{i\omega t} e^{ikx}$ form the orthonormal basis for the solutions of the coupled system. Equation (7) transforms into the set of four linear equations:

$$\begin{cases} (\omega + \beta k)\, a_4 = \delta\,(a_1 - a_3) \\ (\omega - \beta k)\, a_3 = \delta\,(a_2 - a_4) \\ (\omega + \beta k)\, a_2 = -\delta\,(a_1 - a_3) \\ (\omega - \beta k)\, a_1 = -\delta\,(a_2 - a_4) \end{cases} \tag{13}$$

By inspection, we find that $a_1 = a_3$ is a solution. It leads to the eigen values: $\omega = \pm\beta k$ which yield in turn $a_2 = a_4$. The actual values of a_1 and a_2 are arbitrary, that is there are no correlations between the forward and backward directions of propagation of plane waves along the two coupled chains. This solution corresponds to the symmetric mode and will not be addressed further as solution does not exist for the single chain system coupled to a rigid substrate. Another solution of equation (13) can be found when $a_1 = -a_3$ and $a_2 = -a_4$. In that case, the eigen values are $\omega^2 = \beta^2 k^2 + (2\delta)^2$. Inserting $2\delta = +\sqrt{\omega^2 - \beta^2 k^2}$ (the + sign is chosen because δ physically represents a stiffness), into equations (13) yields:

$$\begin{cases} (\omega + \beta k)\, a_4 = \sqrt{\omega^2 - \beta^2 k^2} a_1 \\ (\omega - \beta k)\, a_3 = \sqrt{\omega^2 - \beta^2 k^2} a_2 \\ (\omega + \beta k)\, a_2 = \sqrt{\omega^2 - \beta^2 k^2} a_3 \\ (\omega - \beta k)\, a_1 = \sqrt{\omega^2 - \beta^2 k^2} a_4 \end{cases} \tag{14}$$

Substituting, $\omega + \beta k = \sqrt{\omega + \beta k}\sqrt{\omega + \beta k} = \sqrt{+}\sqrt{+}$, $\omega - \beta k = \sqrt{\omega - \beta k}\sqrt{\omega - \beta k} = \sqrt{-}\sqrt{-}$, and $\sqrt{\omega^2 - \beta^2 k^2} = \sqrt{\omega + \beta k}\sqrt{\omega - \beta k} = \sqrt{+}\sqrt{-} = \sqrt{-}\sqrt{+}$ into equation (14) leads to

$$\begin{cases} \sqrt{+}\sqrt{+}a_4 = \sqrt{+}\sqrt{-}a_1 \\ \sqrt{-}\sqrt{-}a_3 = \sqrt{+}\sqrt{-}a_2 \\ \sqrt{+}\sqrt{+}a_2 = \sqrt{+}\sqrt{-}a_3 \\ \sqrt{-}\sqrt{-}a_1 = \sqrt{+}\sqrt{-}a_4 \end{cases} \tag{15}$$

A possible solution of the set of equations (15) is:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = a_0 \begin{pmatrix} \sqrt{+}\sqrt{+} \\ -\sqrt{+}\sqrt{-} \\ -\sqrt{+}\sqrt{+} \\ \sqrt{+}\sqrt{-} \end{pmatrix} \tag{16}$$

The negative signs in Eq. (16) reflect the antisymmetry of the displacement.

Note also that we could normalize equation (16) by $a_0\sqrt{+}$ to obtain:

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} \sqrt{+} \\ -\sqrt{-} \\ -\sqrt{+} \\ \sqrt{-} \end{pmatrix}$$

In the directional representation using Dirac formalism, we can interpret $\hat{a}_1$ and $\hat{a}_3$ as relating to the amplitude of the forward component of waves propagating in the top and bottom chains in the coupled system. $\hat{a}_2$ and $\hat{a}_4$ relate to the amplitude of the backward component of waves propagating in the top and bottom chains in the coupled system. The displacement of the two coupled harmonic chains are constrained and the direction of propagation of waves in the two-chain system are not independent of each other. For instance, at k=0, the antisymmetric mode is represented by a standing wave with the amplitude of a forward propagating wave and a backward propagating wave being identical. As $k \to \infty$, $\omega \to +\beta k$, the terms $\sqrt{-}$ in equation (16) go to zero and only one direction of propagation is supported by the medium (first and third terms in Eq. (16)). For any other value of the wave number k, the elastic modes supported by the coupled chains are quasi-standing waves which enforce a strict relation between the amplitudes of superposed forward propagating wave and a backward propagating wave. We recall that the constraint on the amplitude of superposed waves forming the antisymmetric modes does not exist for symmetric modes. If one solves the classical Eqs. (1) and (2) using plane wave solutions as is done to obtain Eqs. (4) and (5), one does not find that the solutions are quasi-standing waves with correlation of the amplitudes along the forward and backward directions. Introducing the Dirac formalism (Eq. (7)) enables us to reveal and to project the quasi-standing wave solutions supported by the two coupled chains on forward and backward propagating states.

These amplitudes can be measured directly by measuring the transmission coefficients of the top chain and bottom chain. The measured transmission coefficient of the top chain relates to two of the components of the 4x1 spinor (Eq. 16). Similarly, the transmission coefficient of the bottom chain relates to the other two components. The superposition of elastic states given by equation (16) is therefore measurable without wave function collapse in contrast to what would be the case for the superposition of states of a true quantum system.

In equation (16) a_0 is an arbitrary constant. Because of the relationship between the components of the solution (16), it cannot be expressed, in general, as a tensor product of the solutions of two single chain systems. Table I can be summarized by writing the states of an individual single chain system as $\begin{pmatrix} s_1\sqrt{\pm} \\ s_2\sqrt{\mp} \end{pmatrix}$ with s_1 and s_2 are taking on the values +1 or -1. Here, we express the tensor product of two single chain systems a and b:

$$\begin{pmatrix} s_1^a\sqrt{\pm} \\ s_2^a\sqrt{\mp} \end{pmatrix} \otimes \begin{pmatrix} s_1^b\sqrt{\pm} \\ s_2^b\sqrt{\mp} \end{pmatrix} = \begin{pmatrix} s_1^a s_1^b\sqrt{\pm}\sqrt{\pm} \\ s_1^a s_2^b\sqrt{\pm}\sqrt{\mp} \\ s_2^a s_1^b\sqrt{\mp}\sqrt{\pm} \\ s_2^a s_2^b\sqrt{\mp}\sqrt{\mp} \end{pmatrix} \tag{17}$$

By inspection, one sees that equation (16) cannot always be written in the form of equation (17) but only in a few specific cases.

TABLE I. Spinor components of ψ_k, solutions of equations (10a).

	$e^{+ikx}e^{+i\omega_k t}$	$e^{-ikx}e^{+i\omega_k t}$	$e^{+ikx}e^{-i\omega_k t}$	$e^{-ikx}e^{-i\omega_k t}$
ξ_k	$\begin{pmatrix}\sqrt{\omega+\beta k}\\ \sqrt{\omega-\beta k}\end{pmatrix}$	$\begin{pmatrix}\sqrt{\omega-\beta k}\\ \sqrt{\omega+\beta k}\end{pmatrix}$	$\begin{pmatrix}-\sqrt{\omega-\beta k}\\ \sqrt{\omega+\beta k}\end{pmatrix}$	$\begin{pmatrix}-\sqrt{\omega+\beta k}\\ \sqrt{\omega-\beta k}\end{pmatrix}$

TABLE II. $k \to 0$ and $k \to \infty$ limits of the spinor components of ψ_k.

	$e^{+ikx}e^{+i\omega_k t}$	$e^{-ikx}e^{+i\omega_k t}$	$e^{+ikx}e^{-i\omega_k t}$	$e^{-ikx}e^{-i\omega_k t}$
$\xi_{k\to 0}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$	$\begin{pmatrix}1\\ 1\end{pmatrix}$	$\begin{pmatrix}-1\\ 1\end{pmatrix}$	$\begin{pmatrix}-1\\ 1\end{pmatrix}$
$\xi_{k\to\infty}$	$\begin{pmatrix}1\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 1\end{pmatrix}$	$\begin{pmatrix}0\\ 1\end{pmatrix}$	$\begin{pmatrix}-1\\ 0\end{pmatrix}$

For instance, when $k = 0$, then $\omega = 2\delta$ and $\sqrt{+} = \sqrt{-} = \sqrt{2\delta}$. Equation (16) becomes:

$$\begin{pmatrix}a_1\\ a_2\\ a_3\\ a_4\end{pmatrix} = a_0 2\delta \begin{pmatrix}1\\ -1\\ -1\\ 1\end{pmatrix} = a_0 2\delta \begin{pmatrix}1\\ -1\end{pmatrix} \otimes \begin{pmatrix}1\\ -1\end{pmatrix} \tag{18}$$

When $\delta \to 0$ then $\omega \to \beta k$ and $\sqrt{+} \to \sqrt{2\beta k}$ and $\sqrt{-} \to 0$. Equation (16) reduces to

$$\begin{pmatrix}a_1\\ a_2\\ a_3\\ a_4\end{pmatrix} = a_0\sqrt{2\beta k} \begin{pmatrix}1\\ 0\\ -1\\ 0\end{pmatrix} = a_0\sqrt{2\beta k} \begin{pmatrix}1\\ 0\end{pmatrix} \otimes \begin{pmatrix}1\\ -1\end{pmatrix} \tag{19}$$

The 2x1 vectors $\begin{pmatrix}1\\ -1\end{pmatrix}$ and $\begin{pmatrix}1\\ 0\end{pmatrix}$ are visualized as spinor components of the single chain system.

In summary, the state of the two-chain system is not separable in the tensor product Hilbert space of states of two single chain systems except for the special cases: $k = 0$ and $\delta \to 0$ (or equivalently $k \to \infty$). The choice of this decomposition is driven by the measurability of both the states of single chain systems and two-chain system via measurement of transmission coefficients along the relevant chains.

IV. CONCLUSIONS

We consider a bipartite classical mechanical system composed of two coupled one-dimensional elastic chains whose elastic wave equations can be factored into a Dirac-like equation. Using the Dirac formalism reveals the true nature of elastic waves in this system, namely that they are quasi-standing waves. In the Dirac formalism, the elastic wave functions have spinor components which characterize the strict relation between forward and backward propagating components of quasi-standing waves. This system allows us to address the separability and non-separability of the states of the bipartite elastic system with respect to the choice of elastic subsystems into which it is partitioned. We show that the states of the coupled two-chain system, like any elastic system, can be readily decomposed in the spectral representation. The states of the coupled system span the tensor product space of the energy spectrum associated with wave vector k. The states of the two-chain system are therefore separable into states associated with a spectral representation. In contrast, it is also possible to express the states of the coupled system in the tensor product space of only two subsystems, namely two non-interacting systems composed of a single chain coupled to a rigid substrate. While some of the states of the bipartite system can be written as tensor products of states of the subsystems, there are many states that display correlations that do not allow such decomposition. These states exist in the tensor product Hilbert space of two single chain systems but are not expressible by a simple tensor product. Then, where does the non-separability come from when we can decompose the coupled chains system into two single chain systems. The spectral decomposition describes the orbital part of the wave function. We find new correlations in the directional representation that may lead to non-separability in the spinor part of the amplitude, *i.e.*, correlations between the directions of propagation. The degrees of freedom associated with direction of propagation lead to non-separability. The choice of the decomposition into the tensor product Hilbert space of two single chain systems is motivated by the possibility of measuring directly these degrees of freedom through measurement of transmission coefficients.

To illustrate the applicability of non-separability of the elastic waves, we may consider a two-qubit algorithm,[13] in the spirit of the Deutsch-Jozsa algorithm[14] that exploits non-separability. The intended outcome of this algorithm is to distinguish between the even or odd nature of all possible binary functions for two bit inputs.[13] To identify whether a function is even or odd, one just needs to identify if the final state of the two qubit system is separable or not. The challenge for quantum systems lies now in the measurability of the final states. There is no unambiguous single measurement of entangled states of quantum systems. To distinguish between separable and non-separable superpositions of states, one needs to make multiple measurements and obtain a statistical representation of the superpositions. This drawback could be overcome by using the spinor states of the coupled two-chain system. When the orbital state of the elastic plane wave is defined by $k = 0$ and $\omega = 2\delta$, the spinor state of the system is separable. When the state of the elastic plane wave is defined by a general wave number and frequency, the spinor state of the coupled chains takes a non-separable form. The determination of separable or non-separable elastic states can be done unambiguously via measurement of transmission. In the first case, there is no transmission along the chains and in the second case, transmission can be detected.

The concept of non-separable elastic waves can be extended to three coupled chains, whereby the spinor state of a three mutually coupled chain system is defined in the 2^3-dimensional tensor product Hilbert space of the three single chain systems. Most states of the three-chain system are not separable and measurements of these states can still be achieved via measurement of transmission coefficient along the three chains. Finally, extension to N mutually coupled chains would allow us to conceive the possibility of measureable non-separable spinor states in a 2^N-dimensional space. Such states let us envisage the possibility of storing, processing, and efficiently measuring information in elastic systems with exponential complexity.

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