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Elastic waves with correlated directional and orbital angular momentum degrees of freedom

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Abstract

We introduce a formalism that enables the calculation of elastic wave functions supported by parallel arrays of coupled one-dimensional elastic waveguides. These wave functions are expressed as tensor products of a spinor part associated with directional degrees of freedom and an orbital angular momentum (OAM) part associated with the phase of the coupled waveguides. We demonstrate that one can construct wave functions as a superposition of these elastic waves, which cannot be written as a tensor product of a spinor part and an OAM part. These elastic wave functions are not separable in the tensor product Hilbert space of directional and OAM subspaces. We show that we can construct maximally nonseparable states that are similar to Bell states.

Keywords: elastic waves, orbital angular momentum, nonseparability, phononics

(Some figures may appear in colour only in the online journal)

1. Introduction

The notion of classical 'entanglement' or, in now more generally accepted words, the notion of classical nonseparability [1–3] has been receiving a lot of attention from the theoretical and experimental point of views in the field of optics. Degrees of freedom of photon states that span different Hilbert spaces can be made to interact in a way that leads to local correlations. For instance, laser beams with spin angular momentum and orbital angular momentum (OAM) can be prepared in a nonseparable state [4–11]. Nonseparability of OAM, polarization and radial degrees of freedom of a beam of light has also been achieved [12]. Photonic schemes have also been employed to correlate polarization with propagation direction [13, 14]. To date, the nonseparability between different degrees of freedom has been primarily investigated and demonstrated using laser beams. However, the notion of classical nonseparability is not limited to the optical field but can be applied to other types of excitations such as, for instance, neutron beams [15]. It is the goal of the current paper to demonstrate theoretically, the possibility of achieving correlations between propagation direction and OAM degrees of freedom of elastic waves. More specifically, we show that the elastic waves supported by coupled one-dimensional (1D) elastic waveguides can be described in the tensor product Hilbert space of the direction of propagation and OAM Hilbert subspaces. More importantly, we show that we can construct superpositions of elastic waves in the tensor product space that cannot be factored into tensor products of waves in the subspaces. This signature of nonseparability is completely analogous to that observed in the case of electromagnetic waves. In section 2 of this paper, we present the mathematical formalism that enables us to establish the analogy between superposition of states of elastic waves in multiple coupled waveguide systems and locally correlated multi-degree of freedom systems. Results derived from the model are reported in section 3. We also address the relationship between separability and nonseparability with the notion of measurement of elastic wave transmission amplitudes in section 4. Finally, conclusions regarding the generalization

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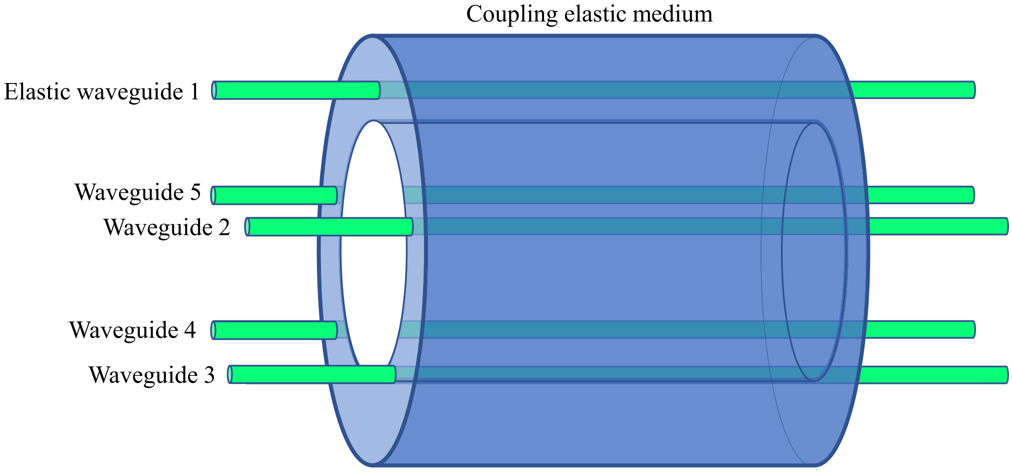

Figure 1. Schematic illustration of elastically coupled one-dimensional waveguides for $N = 5$. The angular separation between the waveguides is $2\pi/N$.

of the notion of wave nonseparability to elastic waves are drawn in section 5.

2. Model and method

We consider a system constituted of N 1D waveguides coupled elastically along their length. The propagation of elastic modes is limited to longitudinal modes along the waveguides in the long wavelength limit i.e., the continuum limit. The equations of motion can be cast into the following compact form:

$$\{H.I_{N\times N} + \alpha^2 M_{N\times N}\}u_{N\times 1} = 0. \tag{1}$$

This system is schematically illustrated in figure 1.

In equation (1), the dynamical differential operator, $H = \frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2}$, models the propagation of elastic waves in the direction x along the waveguides. The parameter β is proportional to the speed of sound in the medium constituting the waveguides implying that the waveguides are constituted of the same material. The parameter α^2 characterizes the strength of the elastic coupling between the waveguides (here we consider that the strength is the same for all coupled waveguides). $u_{N\times 1}$ is a vector which components, u_i, $i = 1, N$, represent the displacement of the ith waveguide. In equation (1), $I_{N\times N}$ is the $N \times N$ identity matrix and the coupling matrix operator $M_{N\times N}$ describes the elastic coupling between waveguides. For instance, in the case of N parallel waveguides in a closed ring arrangement with first neighbor coupling, the coupling matrix is written as the Laplacian matrix with periodic boundary conditions:

$$M_{N\times N} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & -2 \end{pmatrix}. \tag{2}$$

Equation (1) takes the form of a generalized Klein–Gordon (KG) equation. Dirac factorization of the KG equation introduces the notion of the square root of the operator $\{H.I_{N\times N} + \alpha^2 M_{N\times N}\}$. In that factorization, one attempts to represent the dynamics of the system in terms of first derivatives with respect to time, t, and position along the waveguides, x. This factorization reveals the degrees of freedom associated with the direction of propagation of elastic waves, namely the positive or negative directions along the waveguide. There are two possible Dirac equations [16]:

$$\Big\{U_{N\times N}\otimes\sigma_x\frac{\partial}{\partial t} + \beta U_{N\times N}\otimes(-\mathrm{i}\sigma_y)\frac{\partial}{\partial x} \pm \mathrm{i}\alpha U_{2N\times 2N}\sqrt{M_{N\times N}}\otimes\sigma_x\Big\}\Psi_{2N\times 1} = 0. \tag{3}$$

In equation (3), $U_{N\times N}$ and $U_{2N\times 2N}$ are antidiagonal matrices with unit elements. $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_y = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$ are two of the Pauli matrices.$\Psi_{2N\times 1}$ is a $2N$ dimensional vector which represents the modes of vibration of the N waveguides projected in the two possible directions of propagation. $\sqrt{M_{N\times N}}$ is the square root of the coupling matrix. The square root of a matrix is not unique but we will show later that we can pick any form without loss of generality in our search for the elastic modes of the system.

We now choose components of the $\Psi_{2N\times 1}$ vector in the form of plane waves $\psi_I = a_I e^{\mathrm{i}kx}e^{\mathrm{i}\omega t}$ with $I = 1, \ldots, 2N$. k and ω are a wave number and an angular frequency, respectively. Inserting this form in equation (3) and multiplying all terms on the left by $U_{2N\times 2N}$ leads to the Eigen value equation:

$$\{\omega A_{2N\times 2N} + \beta k B_{2N\times 2N} \pm \alpha C_{2N\times 2N}\}a_{2N\times 1} = 0, \tag{4}$$

where

$$A_{2N\times 2N} = I_{N\times N}\otimes I_{2\times 2}, \tag{5a}$$

$$B_{2N\times 2N} = I_{N\times N}\otimes(-\sigma_z), \tag{5b}$$

$$C_{2N\times 2N} = \sqrt{M_{N\times N}}\otimes\sigma_x. \tag{5c}$$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix and $a_{2N\times 1}$ is a $2N$ dimensional vector whose components are the amplitudes a_I.

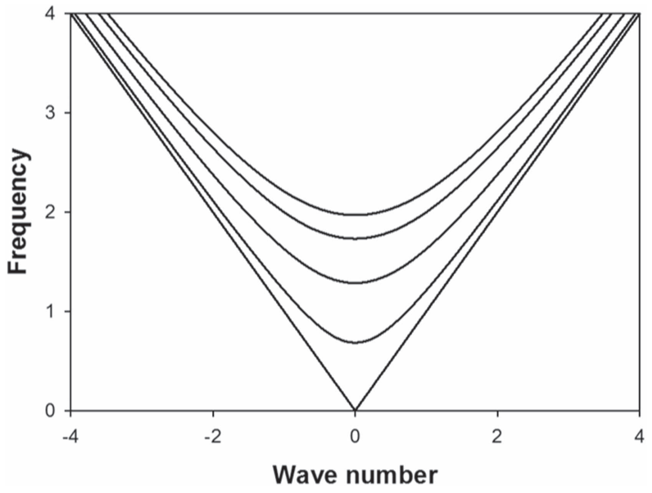

Figure 2. Schematic illustration of the band structure for an array of nine waveguides. The four bands with a cut off frequency are doubly degenerate. We have taken $\beta = 1$ rad s^{-1} and $\alpha = 1$ rad s^{-1}. The wave number is in units of m^{-1} and the frequency is in units of rad s^{-1}.

Equation (4) can be rewritten as a linear combination of tensor products of $N \times N$ and 2×2 matrix operators:

$$\{I_{N\times N} \otimes [\omega I_{2\times 2} - \beta k\sigma_z] \pm \alpha\sqrt{M_{N\times N}} \otimes \sigma_x\}a_{2N\times 1} = 0. \tag{6}$$

This form suggests we seek solutions in the form of tensor products:

$$a_{2N\times 1} = E_{N\times 1} \otimes s_{2\times 1}. \tag{7}$$

The degrees of freedom associated with $E_{N\times 1}$ span an N dimensional Hilbert subspace. The degrees of freedom associated with $s_{2\times 1}$ span a two-dimensional space. In the form of equation (7), the degrees of freedom associated with the coupling of the waveguides and those associated with the propagation along the waveguides are separable into a tensor product and span the tensor product space of the two subspaces.

Replacing $a_{2N\times 1}$ by equation (7) in equation (6) yields:

$$\begin{aligned}&\{(I_{N\times N}E_{N\times 1}) \otimes ([\omega I_{2\times 2} - \beta k\sigma_z]s_{2\times 1})\\ &\pm \alpha(\sqrt{M_{N\times N}}E_{N\times 1}) \otimes (\sigma_x s_{2\times 1})\} = 0.\end{aligned} \tag{8}$$

If we choose $E_{N\times 1}$ to be an Eigen vector, e_n, of the matrix $\sqrt{M_{N\times N}}$ with eigen value λ_n then equation (8) reduces to:

$$e_n \otimes \{([\omega I_{2\times 2} - \beta k\sigma_z] \pm \alpha\lambda_n\sigma_x)s_{2\times 1}\} = 0. \tag{9}$$

For non-trivial eigen vectors e_n, the problem in the space of the directions of propagation reduces to finding solutions of

$$([\omega I_{2\times 2} - \beta k\sigma_z] \pm \alpha\lambda_n\sigma_x)s_{2\times 1} = 0 \tag{10}$$

In obtaining equation (9), we have also used the fact that e_n is an eigen vector of $I_{N\times N}$ with eigen value 1. Equation (9) is the 1D Dirac equation for an elastic system which solutions, $s_{2\times 1}$, have the properties of Dirac spinors [16–19]. The components s_1 and s_2 of the spinor $s_{2\times 1}$ represent the amplitude of the elastic waves in the positive and negative directions, respectively.

We also need to note that the eigen vectors of $\sqrt{M_{N\times N}}$ are also the eigen vectors of the coupling matrix $M_{N\times N}$ itself. Also, the eigen values of $M_{N\times N}$ are λ_n^2. These properties indicate that we do not have to determine the square root of the coupling matrix to find the solutions $a_{2N\times 1}$. One only needs to calculate the eigen vectors and the eigen values of the coupling matrix. Hence, the non-uniqueness of $\sqrt{M_{N\times N}}$ does not introduce difficulties in determining the elastic modes of the coupled system in the Dirac representation.

In the case of a coupling matrix, $M_{N\times N}$, given by equation (2), the eigen values and complex eigen vectors with $n = 0,1, \ldots, N-1$ are obtained as [20]:

$$\lambda_n^2 = 4\left(\sin\frac{\pi n}{N}\right)^2, \tag{11a}$$

$$e_n = \frac{1}{\sqrt{N}}\begin{pmatrix}1\\ e^{i\frac{2\pi n}{N}}\\ \vdots\\ e^{i\frac{2\pi n(N-1)}{N}}\end{pmatrix}. \tag{11b}$$

The real eigen vectors are:

$$e_n = \sqrt{\frac{2}{N}}\begin{pmatrix}1\\ \cos\dfrac{2\pi n}{N}\\ \vdots\\ \cos\dfrac{2\pi n(N-1)}{N}\end{pmatrix} \text{if } \lambda_n \text{ is not degenerate} \tag{12a}$$

$$e_n = \sqrt{\frac{2}{N}}\begin{pmatrix}1\\ \cos\dfrac{2\pi n}{N}\\ \vdots\\ \cos\dfrac{2\pi n(N-1)}{N}\end{pmatrix}$$

$$\text{or}\sqrt{\frac{2}{N}}\begin{pmatrix}0\\ \sin\dfrac{2\pi n}{N}\\ \vdots\\ \sin\dfrac{2\pi n(N-1)}{N}\end{pmatrix} \text{if } \lambda_n \text{ is degenerate.} \tag{12b}$$

For instance, three coupled chains lead to two eigen values $\lambda_0^2 = 0$ and $\lambda_1^2 = 3$. The first eigen value is not degenerate but the later eigen value is doubly degenerate. In the case of four waveguides, we have $\lambda_0^2 = 0$, $\lambda_1^2 = 2$ and $\lambda_3^2 = 4$. The first and third modes are not degenerate. The second mode is doubly degenerate. There are three distinct eigen values for five coupled chains with only the $\lambda_0^2 = 0$ being non-degenerate. Systems with different odd and even number of chains N possess modes with different degeneracy.

The operator $M_{N\times N}$, its eigen values and eigen vectors are consistent with the notion of OAM of elastic waves propagating along the coupled waveguides. The components of the eigen vectors depend on the angular position along the coupled waveguides ring arrangement. The elastic waves propagating in the coupled waveguide system exhibit helical phase fronts. The various OAM eigen vectors include monopolar, dipolar, quadrupolar, etc modes.

3. Results

Equation (10) can now be solved for a given λ_n. In matrix form it becomes:

$$\begin{pmatrix} \omega_n - \beta k & \pm\alpha\lambda_n \\ \pm\alpha\lambda_n & \omega_n + \beta k \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0. \tag{13}$$

This eigen equation gives the dispersion relation $\omega_n^2 = (\beta k)^2 + (\alpha\lambda_n)^2$ (see figure 2) and the following eigen vectors projected into the space of directions of propagation:

$$s_{2\times 1} = s_0\begin{pmatrix} \sqrt{\omega_n + \beta k} \\ \pm\sqrt{\omega_n - \beta k} \end{pmatrix}. \tag{14}$$

Note that this dispersion relation support both negative and positive frequency solutions. In equation (14), s_0 is a complex constant. The amplitudes in the positive and negative directions along the waveguides are not independent unless $n = 0$. When $\lambda_0^2 = 0$, propagation along the two directions decouple and the dispersion relations reduces to that of a homogeneous medium, namely $\omega = \pm\beta k$. For $n \neq 0$, the dispersion relation shows a cut off frequency which depends on the eigen value of the OAM degree of freedom. The amplitude in the two possible directions of propagation are dependent on the wave number, k. These correspond physically to quasistanding waves with full standing wave character for $k = 0$ and full traveling character for $k \to \infty$.

For a degenerate band, the complete solution for the elastic field in the coupled structure is now given by (equation (7)):

$$a_{2N\times 1} = e_n \otimes s_{2\times 1} = \sqrt{\frac{2}{N}}\begin{pmatrix} 1 \\ \cos\frac{2\pi n}{N} \\ \vdots \\ \cos\frac{2\pi n(N-1)}{N} \end{pmatrix} \otimes s_0\begin{pmatrix} \sqrt{\omega_n + \beta k} \\ \pm\sqrt{\omega_n - \beta k} \end{pmatrix} = \sqrt{\frac{2}{N}}s_0 \times \begin{pmatrix} \sqrt{\omega_n + \beta k} \\ \pm\sqrt{\omega_n - \beta k} \\ \sqrt{\omega_n + \beta k}\cos\frac{2\pi n}{N} \\ \pm\sqrt{\omega_n - \beta k}\cos\frac{2\pi n}{N} \\ \vdots \\ \sqrt{\omega_n + \beta k}\cos\frac{2\pi n(N-1)}{N} \\ \pm\sqrt{\omega_n - \beta k}\cos\frac{2\pi n(N-1)}{N} \end{pmatrix}. \tag{15}$$

Since equation (3) is linear, any superposition of modes is also a solution of equation (3). Also as the system is doubly degenerate in momentum space, one can for instance construct a superposition of two waves with the same frequency but positive and negative wave numbers for a doubly degenerate band n:

$$\Psi_{2N\times 1} = \sqrt{\frac{2}{N}}\begin{pmatrix} 1 \\ \cos\frac{2\pi n}{N} \\ \vdots \\ \cos\frac{2\pi n(N-1)}{N} \end{pmatrix} \otimes s_0^{(k)}\begin{pmatrix} \sqrt{\omega_n + \beta k} \\ \pm\sqrt{\omega_n - \beta k} \end{pmatrix} \times e^{ikx}e^{i\omega_n t} + \sqrt{\frac{2}{N}}\begin{pmatrix} 0 \\ \sin\frac{2\pi n}{N} \\ \vdots \\ \sin\frac{2\pi n(N-1)}{N} \end{pmatrix} \otimes s_0^{(-k)}\begin{pmatrix} \sqrt{\omega_n - \beta k} \\ \pm\sqrt{\omega_n + \beta k} \end{pmatrix} e^{-ikx}e^{i\omega_n t}. \tag{16}$$

In this superposition, both the eigen vectors of the OAM degree of freedom and the spinors are different. This superposition of states cannot be written in the form of a tensor product of one OAM eigen vector, $D_{N\times 1}$ and one spinor, $s'_{2\times 1}$. If that were the case, we could say that:

$$\Psi_{2N\times 1} = D_{N\times 1} \otimes s'_{2\times 1}e^{ik'x}e^{i\omega' t}, \tag{17}$$

where k' and ω' are some wave number and frequency. From equations (17) and (16) it is clear that one must have $\omega' = \omega_n$. Furthermore, equating the ratio of two successive odd and even components with index J of the tensor product in equation (17) and of the wave function given by equation (16) yields:

$$\frac{\gamma^+ e^{ikx} + \gamma^- e^{-ikx}}{\delta^+ e^{ikx} + \delta^- e^{-ikx}} = \frac{s_1'}{s_2'} \tag{18}$$

with $\gamma^+ = \cos\frac{2\pi nJ}{N}\sqrt{\omega_n + \beta k}$, $\gamma^- = \sin\frac{2\pi nJ}{N}\sqrt{\omega_n - \beta k}$, $\delta^+ = \cos\frac{2\pi nJ}{N}\sqrt{\omega_n - \beta k}$, and $\delta^- = \sin\frac{2\pi nJ}{N}\sqrt{\omega_n + \beta k}$. Here we have taken the + spinor component of the $\pm$ in equation (16). Equation (18) can be reformulated as:

$$Pe^{ikx} + Qe^{ikx} = 0, \tag{19}$$

where $P = s_2'\gamma^+ - s_1'\delta^+$ and $Q = s_2'\gamma^- - s_1'\delta^-$ are real quantities.

Equation (19) must be satisfied for all x and, in particular, for $x = 0$. At the origin, we have $P = -Q$ which when inserted into equation (19) results in $P(-2i\sin kx) = 0$. The condition for separability of $\Psi_{2N\times 1}$ into a tensor product of a OAM eigen vector and a spinor requires $k = 0$. This corresponds to pure standing waves. All other superpositions of quasistanding waves of the form given by equation (16) are not separable. These superpositions span the tensor product Hilbert space of the N-dimensional OAM subspace and the two-dimensional direction of propagation subspace but cannot be written as tensor products in the $2N$-dimensional Hilbert space.

We now denote by $e_n^{(1)}$ and $e_n^{(2)}$ the wave functions associated with the OAM degree of freedom for a doubly degenerate band n. We also denote by $f_k = \frac{1}{\sqrt{2\omega_n}}\begin{pmatrix}\sqrt{\omega_n + \beta k}\\ \pm\sqrt{\omega_n - \beta k}\end{pmatrix}e^{ikx}$ and $f_{-k} = \frac{1}{\sqrt{2\omega_n}}\begin{pmatrix}\sqrt{\omega_n - \beta k}\\ \pm\sqrt{\omega_n + \beta k}\end{pmatrix}e^{-ikx}$, the normalized spinorial part of the wave functions associated with the directional degrees of freedom. We can form a basis for the states of the coupled elastic waveguides in the form of the four tensor products:

$\phi_1 = e_n^{(1)} \otimes f_k$; $\phi_2 = e_n^{(1)} \otimes f_{-k}$; $\phi_3 = e_n^{(2)} \otimes f_k$; $\phi_4 = e_n^{(2)} \otimes f_{-k}$. In that basis the state giving by equation (16) reads:

$$\Psi_{2N\times 1} = \sqrt{2\omega_n}\,(s_0^{(k)} e_n^{(1)} \otimes f_k + s_0^{(-k)} e_n^{(2)} \otimes f_{-k}) \times e^{i\omega_n t} = \sqrt{2\omega_n}\,(s_0^{(k)}\phi_1 + s_0^{(-k)}\phi_4)e^{i\omega_n t}. \quad (20)$$

It is clear that this state cannot be written as a tensor product in the basis $\{\phi_1, \phi_2, \phi_3, \phi_4\}$. Again, we can say that this state is nonseparable. To quantify the degree of nonseparability of this state, we can calculate the 'entanglement entropy' [21]. Here we use the term 'entanglement' in quotation marks to stress the classical nature of nonseparability. First, we normalize the wave function of equation (20):

$$\tilde{\Psi} = \frac{1}{\sqrt{(|s_0^{(k)}|^2 + |s_0^{(-k)}|^2)}}(s_0^{(k)}\phi_1 + s_0^{(-k)}\phi_4)e^{i\omega_n t}. \quad (21)$$

In obtaining the normalizing factor in equation (21) we have used the fact that $\phi_i\phi_j^* = (e_n^{(i)} \otimes f_{\pm k})(e_n^{(j)} \otimes f_{\pm k})^* = (e_n^{(i)} \otimes f_{\pm k})(e_n^{(i)*} \otimes f_{\pm k}^*) = (e_n^{(i)} e_n^{(i)*}) \otimes (f_{\pm k} f_{\pm k}^*)$ and that the vectors $e_n^{(i)}$ (equation (12b)) form an orthonormal basis. We also note that the amplitude of $f_{\pm k}$ is real.

We can also define the basis $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ in terms of the 4×1 vectors $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$. We construct the density matrix associated with that state as the outer product of $\tilde{\Psi}$ and its complex conjugate $\tilde{\Psi}^*$:

$$\rho_{\mathrm{OAM}-s} = \tilde{\Psi} \otimes \tilde{\Psi}^* = \frac{1}{|s_0^{(k)}|^2 + |s_0^{(-k)}|^2} \times \begin{pmatrix} s_0^{(k)}s_0^{(k)*} & 0 & 0 & s_0^{(k)}s_0^{(-k)*}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ s_0^{(-k)}s_0^{(k)*} & 0 & 0 & s_0^{(-k)}s_0^{(-k)*}\end{pmatrix}. \quad (22)$$

In equation (22), we have used the notation $\rho_{\mathrm{OAM}-s}$ to highlight that the density of states is expressed in the tensor product Hilbert space of the directional and OAM subspaces. The reduced density of state in the Hilbert space of OAM is obtained by taking the partial trace of the density matrix over the directional states:

$$\rho_{\mathrm{OAM}} = \frac{1}{|s_0^{(k)}|^2 + |s_0^{(-k)}|^2}\begin{pmatrix} s_0^{(k)}s_0^{(k)*} & 0\\ 0 & s_0^{(-k)}s_0^{(-k)*}\end{pmatrix}. \quad (23)$$

The entropy of 'entanglement' is now obtained from the relation:

$$S(\rho_{\mathrm{OAM}}) = -Tr(\rho_{\mathrm{OAM}} \ln \rho_{\mathrm{OAM}}). \quad (24)$$

Using equation (23), we get:

$$S(\rho_{\mathrm{OAM}}) = -\frac{1}{|s_0^{(k)}|^2 + |s_0^{(-k)}|^2}\left(s_0^{(k)}s_0^{(k)*}\ln \times \frac{s_0^{(k)}s_0^{(k)*}}{(s_0^{(k)})^2 + (s_0^{(-k)})^2} + s_0^{(-k)}s_0^{(-k)*}\ln \times \frac{s_0^{(-k)}s_0^{(-k)*}}{(s_0^{(k)})^2 + (s_0^{(-k)})^2}\right). \quad (25)$$

We can also calculate the entropy of 'entanglement' by calculating the reduced density matrix in the Hilbert space of directions, ρ_s. The two entropies are equal.

We note that if one chooses $s_0^{(k)} = 1$ and $s_0^{(-k)} = 1$, $S(\rho_{\mathrm{OAM}}) = \ln 2$. The state $\tilde{\Psi}$ (equation (21)) is maximally 'entangled.' The state $\tilde{\Psi} = \frac{1}{\sqrt{2}}(\phi_1 + \phi_4)e^{i\omega_n t}$ is equivalent to a Bell state [20].

By controlling the amplitude and phase of $s_0^{(k)}$ and $s_0^{(-k)}$, one can control the degree of nonseparability of the $\tilde{\Psi}$ as well as generate other Bell states.

Other examples of nonseparable states can also be obtained by considering superpositions of modes in two different bands (each possessing a cut off frequency), n and n', with the same frequency $\omega_n = \omega_{n'}$ but different wave numbers, k and k':

$$\Psi'_{2N\times 1} = \sqrt{\frac{2}{N}}\begin{pmatrix}1\\ \cos\frac{2\pi n}{N}\\ \vdots\\ \cos\frac{2\pi n(N-1)}{N}\end{pmatrix} \otimes s_0^{(k)}\begin{pmatrix}\sqrt{\omega_n + \beta k}\\ \pm\sqrt{\omega_n - \beta k}\end{pmatrix} \times e^{ikx}e^{i\omega t} + \sqrt{\frac{2}{N}}\begin{pmatrix}1\\ \cos\frac{2\pi n'}{N}\\ \vdots\\ \cos\frac{2\pi n'(N-1)}{N}\end{pmatrix} \otimes s_0^{(k')}\begin{pmatrix}\sqrt{\omega_{n'} - \beta k'}\\ \pm\sqrt{\omega_{n'} + \beta k'}\end{pmatrix}e^{ik'x}e^{i\omega_{n'}t}. \quad (26)$$

In this case the frequency is limited to the range defined by the intersection of the frequency range of the two bands, n and n'.

4. Discussion

Let us discuss the meaning of the notions of separability and nonseparability of elastic states in the context of manipulation of elastic states and measurement. For practical reasons, we would have to work with a finite length array of coupled elastic waveguides. The modes will now be discrete in the directional degrees of freedom. Let us consider a separable state of the form $e_n \otimes s_{2\times1}e^{ikx}e^{i\omega t}$. This mode can be excited with N transducers attached to the input ends of the N waveguides and connected to N phase-locked signal generators to excite the appropriate OAM mode. The frequency is used to control the spinor state. One can measure the directional degrees of freedom independently of the OAM degrees of freedom. The spinor part of the wave function $s_{2\times1}$ is the same for each waveguide. The components of $s_{2\times1}$ which represent a quasistanding wave can be quantified by measuring the transmission coefficient (normalized transmitted amplitude) along any one of the waveguides. It is then possible to operate on the OAM without affecting the spinor state. For instance, one could apply a rotation that permutes cyclically the components of e_n by changing the phase of the signal generators. Such an operation could be quantified by measuring the phase of the transmission amplitude at the output end of the waveguides. In contrast, a nonseparable state such as that given by equation (26), can be excited by applying a superposition of signals on the transducers with the appropriate phase, amplitude and frequency relations. However, because this state is not separable, measurements of transmission amplitude and phases along each waveguide cannot be separated into independent spinor and OAM parts. Any operation such as the application of a rotation to the OAM degrees of freedom will result in a change in the spinorial character of every waveguide.

5. Conclusions

The present work shows that the state of elastic waves supported by a parallel array of N elastically coupled waveguides can be characterized by two types of degrees of freedom, namely an OAM degree of freedom spanning a N-dimensional subspace and the direction of propagation spanning a two-dimensional subspace. The states of elastic waves in this system are spanning the tensor product Hilbert space of these two subspaces. It is possible to construct linear combinations of these states that cannot be expressed as a tensor product in the larger Hilbert space. These states are said to be nonseparable and are therefore locally correlated. Here we have considered an array of waveguides that form a ring-like arrangement. The mathematical framework developed in the present work is general and can be applied to any other arrangement of the coupled waveguides. The arrangement will change the form of the coupling matrix and therefore the eigen values and eigen vectors in the N-dimensional subspace. The direction of propagation degrees of freedom will remain unchanged. The overall conclusion regarding the existence of nonseparable superpositions of elastic states will not be affected by the details of the arrangements. Furthermore, fully two-dimensional arrays of waveguides such as concentric ring arrangements will lead to the introduction of another degree of freedom related to the position of the waveguide (e.g., radial position with respect to the origin). One can anticipate the possibility of creating elastic beams along this type of array with correlated directional/OAM and radial position degrees of freedom. The ability to produce light beams supporting correlated OAM/polarization/radial degrees of freedom has allowed the development of useful techniques with applications to photonic-based quantum information science. The extension of the notion of nonseparability to elastic waves and the demonstration of the analogy between elastic wave propagation and quantum mechanics opens unique opportunities in the emerging field of phononics.

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References

[1] Spreeuw R J 1998 A classical analogy of entanglement *Found. Phys.* **28** 361
[2] Ghose P and Mukherjee A 2014 Entanglement in classical optics *Rev. Theor. Sci.* **2** 274
[3] Karimi E and Boyd R W 2015 Classical entanglement? *Science* **350** 1172
[4] Souza C E R, Huguenin J A O, Milman P and Khoury A Z 2007 Topological phase for spin–orbit transformations on a laser beam *Phys. Rev. Lett.* **99** 160401
[5] Chen L and She W 2010 Single-photon spin–orbit entanglement violating a Bell-like inequality *J. Opt. Soc. Am.* B **27** A7
[6] Borges C V S, Hor-Meyll M, Huguenin J A O and Khoury A Z 2010 Bell-like inequality for the spin–orbit separability of a laser beam *Phys. Rev.* A **82** 033833
[7] Karimi E, Leach J, Slussarenko S, Piccirillo B, Marrucci L, Chen L, She W, Franke-Arnold S, Padgett M J and Santamato E 2010 Spin–orbit hybrid entanglement of photons and quantum contextuality *Phys. Rev.* A **82** 022115
[8] Valles A, D'Ambrosio V, Hendrych M, Micuda M, Marrucci L, Sciarrino F and Torres J P 2014 Generation of tunable entanglement and violation of a Bell-like inequality between different degrees of freedom of a single photon *Phys. Rev.* A **90** 052326
[9] Pereira L J, Khoury A Z and Dechoum K 2014 Quantum and classical separability of spin–orbit laser modes *Phys. Rev.* A **90** 053842
[10] Qian X-F, Little B, Howell J C and Eberly J H 2015 Shifting the quantum–classical boundary: theory and experiment for statistically classical optical fields *Optica* **2** 611
[11] Balthazar W F, Souza C E R, Caetano D P, Galvão E F, Huguenin J A O and Khoury A Z 2016 Tripartite nonseparability in classical optics *Opt. Lett.* **41** 5797
[12] Rafsanjani S M H, Mirhosseini M, Magana-Loaiza O S and Boyd R W 2015 State transfer based on classical nonseparability *Phys. Rev.* A **92** 023827
[13] Michler M, Weinfurter H and Żukowski M 2000 Experiments towards falsification of noncontextual hidden variable theories *Phys. Rev. Lett.* **84** 5457

[14] Gadway B, Galvez E and Zela F D 2008 Bell-inequality violations with single photons entangled in momentum and polarization *J. Phys. B: At. Mol. Opt. Phys.* **42** 015503
[15] Hasegawa Y, Loidl R, Badurek G, Baron M and Rauch H 2003 Violation of a Bell-like inequality in single-neutron interferometry *Nature* **425** 45
[16] Deymier P A, Runge K, Swinteck N and Muralidharan K 2014 Rotational modes in a phononic crystal with fermion-like behavior *J. Appl. Phys.* **115** 163510
[17] Deymier P A, Runge K, Swinteck N and Muralidharan K 2015 Torsional topology and fermion-like behavior of elastic waves in phononic structures *C. R. Acad. Sci., Paris* II **343** 700–11
[18] Deymier P A and Runge K 2016 One-dimensional mass-spring chains supporting elastic waves with non-conventional topology *Crystals* **6** 44
[19] Deymier P A and Runge K 2017 Sound topology, duality, coherence and wave-mixing: an introduction to the emerging new science of sound *Springer Series in Solid State Sciences* vol 188 (Berlin: Springer)
[20] Othmer H G and Scriven L E 1971 Instability and dynamic pattern in cellular networks *J. Theor. Biol.* **32** 507n
[21] Janzing D 2009 Entropy of entanglement *Compendium of Quantum Physics* ed D Greenberger, K Hentschel and F Weinert (Berlin: Springer) pp 205–9