Journal of Physics Communications

PAPER • OPEN ACCESS

Exponentially complex nonseparable states in planar arrays of nonlinearly coupled one-dimensional elastic waveguides

To cite this article: P A Deymier *et al* 2020 *J. Phys. Commun.* **4** 085018

IOP Publishing J. Phys. Commun. 4 (2020) 085018 https://doi.org/10.1088/2399-6528/abb0f0

Journal of Physics Communications

CrossMark

OPEN ACCESS

RECEIVED
22 May 2020

REVISED
13 August 2020

ACCEPTED FOR PUBLICATION
20 August 2020

PUBLISHED
1 September 2020

PAPER

Exponentially complex nonseparable states in planar arrays of nonlinearly coupled one-dimensional elastic waveguides

P A Deymier[1], K Runge and M Arif Hasan

Department of Materials Science and Engineering, The University of Arizona, Tucson, Arizona 85721, United States of America

[1] Author to whom any correspondence should be addressed.

E-mail: deymier@email.arizona.edu, krunge@email.arizona.edu and mdhasan@email.arizona.edu

Keywords: Nonlinear elasticity, nonseparabke elastic states, exponentially complex Hilbert space

Abstract

Externally driven parallel arrays of elastic waveguides inelastically coupled along their length are shown to support inelastic modes that span exponentially complex Hilbert spaces. The nonlinear coupling takes the form of power functions of the relative displacement between waveguides. First order perturbation theory is employed to obtained closed form solutions for the contribution of nonlinearity to the displacement field in the form of nonseparable superpositions of product states of plane waves. When the system is dissipative, the coefficients of the superposition of product states are complex and can be varied by controlling the characteristics of the way the array is driven externally. The entropy of 'entanglement' of these nonseparable superpositions of product states is calculated from the complex amplitudes through the density matrix. Navigation of the exponentially complex Hilbert space of product states enables manipulation of the degree of nonseparability of the superpositions.

1. Introduction

Superposition of states and entanglement are at the core of today's second quantum revolution [1]. Entangled superpositions of states possess two distinct attributes, namely nonlocality and nonseparability. Nonlocality is a unique feature of quantum mechanics, but nonseparability is not. The notion of 'classical entanglement', i.e., local nonseparable superposition of states has received a significant amount of attention, theoretically and experimentally, in the area of optics [2–4]. For instance, degrees of freedom of photon states that span different Hilbert spaces have been locally correlated. Laser beams with spin angular momentum and orbital angular momentum (OAM) can be prepared in a nonseparable state [5–12]. Nonseparability of OAM, polarization and radial degrees of freedom of a beam of light has also been reported [13]. Photonic schemes have also been employed to correlate polarization with propagation direction [14, 15]. The notion of classical nonseparability has also been extended beyond the optical field to neutron beams [16]. However, to date, much less attention has been paid to elastic waves; yet, remarkable new behaviors of sound, analogous to quantum physics, such as the notions of elastic pseudospin [17–22] and Zak/Berry phase [23–29], are emerging. Recently, elastic nonseparable states, analogous to 'classically entangled' states were observed experimentally in externally driven systems composed of parallel arrays of one-dimensional elastic waveguides coupled elastically along their length [30]. These classically nonseparable states are Bell states constructed as a superposition of elastic waves, each a product of a plane wave part spanning the length of the waveguides and a spatial part spanning the direction across the array of waveguides. We demonstrated that the amplitude coefficients of the nonseparable superposition of states are complex due to dissipation in the constitutive elastic materials. By tuning these complex amplitudes, we have shown that we can experimentally navigate a sizeable portion of the Bell state's Hilbert space. These states lie in the tensor product Hilbert space of the subspaces associated with the degrees of freedom along and across the waveguide array. The dimension of this product space scales linearly with the number of waveguides as $2N$ where N is the number of waveguides. In order to achieve the full potential of the second quantum revolution, it is highly desirable to construct classical nonseparable states that lie in an

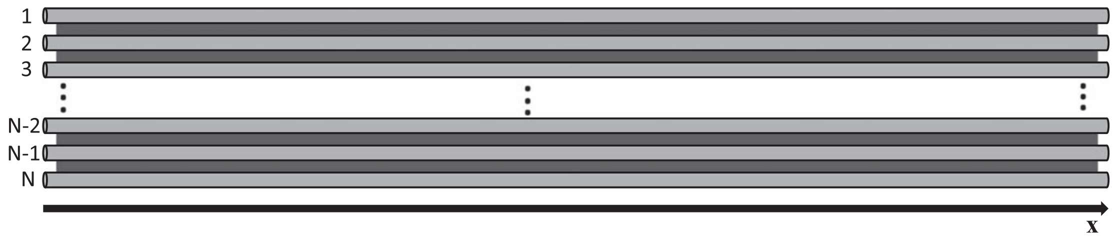

Figure 1. Schematic representation of N one-dimensional waveguides (rods) coupled along their length.

exponentially complex Hilbert space. More specifically, one would like to design a multipartite elastic system composed of Q subsystems or degrees of freedom, each of which able to be in at least two states. The dimension of the Hilbert space of such a system would then take the value 2^Q which scales exponentially with the number Q. We have recently shown such a behavior in a system composed of one-dimensional elastic waveguides which were coupled via quadratic nonlinear forces along their length [31]. The elastic nonlinearity by enabling wave-wave interaction was necessary to achieve wave mixing and therefore to allow the formation of waves which frequency and wave number are the sum of the frequencies and wave numbers of parent linear waves. These nonlinear waves were product waves of the parent waves and therefore spanned a 2^2 dimensional Hilbert space which was the product of the spaces supporting the parent waves (each one two-dimensional).

In the present paper, we generalize this approach to planar array of $N \geqslant 3$ nonlinearly coupled waveguides with nonlinear exponent $Q < N$ driven externally by external harmonic forces. We show that to first order in perturbation, if we excite Q of the N possible bands and two plane wave states in each band, the nonlinear elastic system can be visualized as a multipartite two level system which can support superpositions of nonlinear modes which span exponentially complex Hilbert spaces. By partitioning the spatial degrees of freedom across the planar array, i.e., separating the system into two subsystems, we implement an approach based on the density matrix of the subsystems to calculate the entropy of entanglement. This approach is extended to all values of N and Q by the use of singular value decompositions which leads to the possibility of achieving maximally entangled exponentially complex superpositions of nonlinear product states.

The paper is organized as follows. In section 2 we introduce a formalism that enables us to determine the acoustic field of any externally driven planar array of one-dimensional waveguides coupled inelastically along their length. The nonlinear coupling can be any power, Q, of the displacement as long as it is smaller than the number of waveguides. This formalism is based on first order perturbation theory. Section 3 presents three examples with $N = 3,\ 4,\ 5$ and $Q = N - 1$. We present the nonlinear component of the displacement field as a superposition of product of plane waves. Section 4 introduces the method for the calculation of the entanglement entropy which is then applied to the example systems of section 3. Finally, we draw some conclusions as to the applicability of driven planar arrays of nonlinearly coupled one-dimensional elastic waveguides supporting exponentially complex nonseparable states as quantum analogues in the area of quantum information science.

2. Planar array of N parallel externally driven waveguides coupled elastically along their length

We consider N one-dimensional elastic waveguides coupled elastically along their length. The system is driven externally at some position $x = 0$ (figure 1).

In the long wavelength limit the wave equation takes the form:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t}\right)\vec{I} + \alpha^2\overleftrightarrow{M}\right]\vec{U} = \vec{F}\delta_{x=0}e^{i\omega_d t} \tag{1}$$

The parameter β is proportional to the speed of sound in the waveguides. x represents the position along the waveguides. μ is a damping parameter. $\vec{I}$ is the identity matrix. α represents the coupling strength between waveguides with $\overleftrightarrow{M}$ being the matrix describing the elastic coupling between the N waveguides. $\vec{F}$ represents the driving force. In the case of a planar array of waveguides, the coupling matrix takes the form:

$$\overleftrightarrow{M} = \begin{pmatrix} 1 & -1 & 0 & 0 & . & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & . & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & . & 0 & 0 & -1 & 1 \end{pmatrix}$$

The $N \times 1$ vector $\vec{U} = (U_1, U_2, \ldots, U_N)$ represents the displacement in waveguides 1 through N, respectively. In equation (1), ω_d is the frequency of the driver.

We now define λ_n and $\vec{E}_n$ with $n = 1, 2, \ldots, N$, the eigen values and eigen vectors of the matrix $\overleftrightarrow{M}$. The $\vec{E}_n$ represent the spatial eigen modes across the waveguides with components $E_{n,j}$, with the jth rod in the nth mode. We can write:

$$\overleftrightarrow{M}\vec{E}_n = \lambda_n \overleftrightarrow{I}\ \vec{E}_n \tag{2}$$

Since the $\vec{E}_n$ form a complete orthonormal basis, we can write the displacement vector as:

$$\vec{U} = \sum_n u_n \vec{E}_n \tag{3}$$

The $N \times 1$ vector, $\vec{F}$, is also expressed in the $\vec{E}_n$ basis:

$$\vec{F} = \sum_n F_n \vec{E}_n \tag{4}$$

The F_n's are therefore defined as the dot product $\vec{F}.\vec{E}_n$ between the two $N \times 1$ vectors.

Inserting equations (2)–(4) in equation (1) leads to a set of N equations of the form:

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t} + \alpha^2\lambda_n\right)u_n = F_n\delta_{x=0}e^{i\omega_d t} \tag{5}$$

We seek plane wave solutions which follow the driver in time:

$$u_n = \sum_{k_n} A_n(k_n)e^{ik_n x}e^{i\omega_d t} \tag{6}$$

k_n is a wave number.

Inserting equation (6) into equation (5) evaluated at $x = 0$ yields the driven complex amplitudes:

$$A_n(k_n) = \frac{F_n}{\omega_{0,n}^2(k_n) - \omega_d^2 + i\mu\omega} \tag{7}$$

where the characteristic frequency

$$\omega_{0,n}^2(k_n) = \beta^2 k_n^2 + \alpha^2\lambda_n \tag{8}$$

The complete displacement field is therefore obtained as

$$\vec{U} = \sum_{N}^{n=1} \vec{E}_n \sum_{k_n} A_n(k_n)e^{ik_n x}e^{i\omega_d t} \tag{9}$$

with the complex resonant amplitudes given by equation (7). In equations (6) and (9), the summation over the wave numbers, k_n, is discrete, thus implying that the waveguides have the same finite length. For a given driving frequency, we now limit the summation over the wave numbers to the two terms which contribute the most to the displacement field. Let k_n and k_n' be the wave numbers for which $\omega_{0,n}^2 - \omega_d^2$ is small. This is illustrated in figure 2.

We can now approximate equation (9) by:

$$\vec{U} = \sum_{n=1}^{N} \vec{E}_n(A_n(k_n)e^{ik_n x} + A_n'(k_n')e^{ik_n' x})e^{i\omega_d t} \tag{10}$$

We note that the number of terms in the sum, Σ_n, can be reduced by nullifying some of the complex amplitudes (i.e. choosing some of the F_n to be zero).

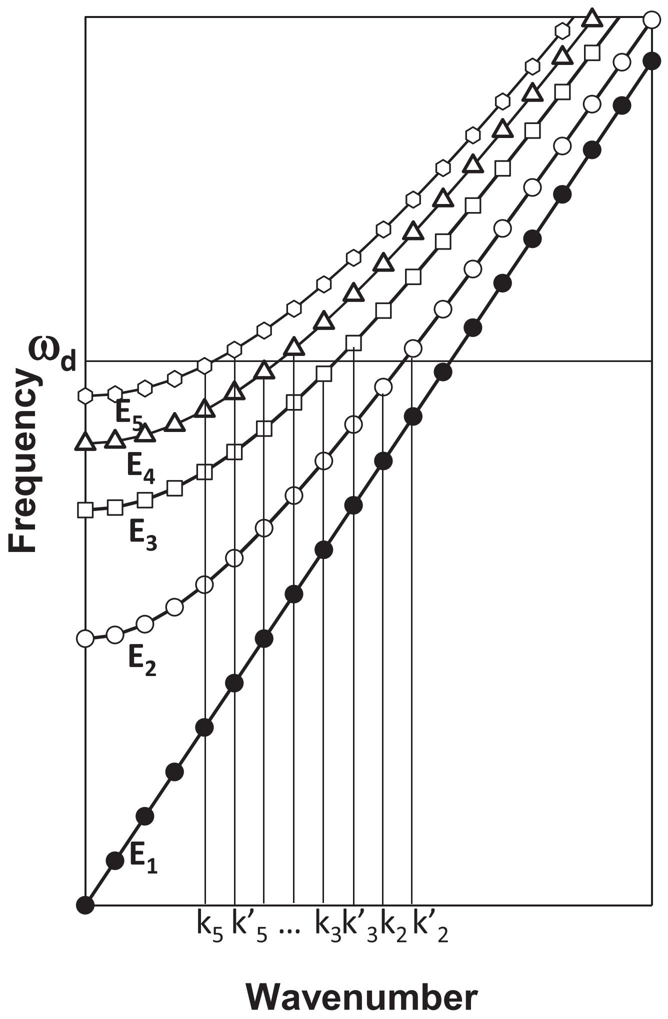

Figure 2. Schematic representation of the band structure of an array of N = 5 waveguides. The spatial modes are labelled E_n, $n = 1,..,5$. The k_n and k_n' are the wave numbers of the plane waves contributing the most to the complex amplitudes given by equation (7) at the driving frequency ω_d.

3. Planar array of *N* parallel externally driven waveguides coupled inelastically along their length

We introduce a nonlinear coupling term in equation (1):

$$\left[\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t}\right)\vec{I} + \alpha^2\vec{M}\right]\vec{U} + \varepsilon\vec{G} = \vec{F}\delta_{x=0}e^{i\omega_d t} \tag{11}$$

where

$$G_1 = (U_2 - U_1)^Q \tag{12a}$$

$$G_{1<j<N} = (U_{j-1} - U_j)^Q + (U_{j+1} - U_j)^Q \tag{12b}$$

$$G_N = (U_{N-1} - U_N)^Q \tag{12c}$$

We assume that the exponent of the nonlinear terms is an integer, $N - 1 \geqslant Q > 2$.

We solve equation (11) within perturbation theory. Assuming a small ε, we expand the displacement field to first order in perturbation:

$$\vec{U} = \vec{U}^{(0)} + \varepsilon\vec{U}^{(1)} \tag{13}$$

Inserting equation (13) into equation (11) yields to zeroth order in perturbation:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t}\right)\vec{I} + \alpha^2\vec{M}\right]\vec{U}^{(0)} = \vec{F}\delta_{x=0}e^{i\omega_d t} \tag{14}$$

Equation (14) is nothing but the linear equation (1) which solutions, $\vec{U}^{(0)}$, are given by equation (10).

To first order in ε we obtain:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t}\right)\vec{I} + \alpha^2\vec{M}\right]\vec{U}^{(1)} = -\vec{G}^{(0)} \tag{15}$$

$\vec{G}^{(0)}$ is given by equations (12a)–(12c) with the displacement $\vec{U}^{(0)}$. In equation (15), the zeroth order displacement drives the system.

Let us consider the nonlinear terms of the form:

$$\begin{aligned}(U_{j+1} - U_j)^Q &= \left(\sum_n^* E_{n,j+1}(A_n e^{ik_n x} + A_n' e^{ik_n' x}) - \sum_n^* E_{n,j}(A_n e^{ik_n x} + A_n' e^{ik_n' x})\right)^Q e^{iQ\omega_d t}\\ &= \left(\sum_n^*(E_{n,j+1} - E_{n,j})(A_n e^{ik_n x} + A_n' e^{ik_n' x})\right)^Q e^{iQ\omega_d t}\end{aligned} \tag{16}$$

In equation (16) we have chosen the driving force $\vec{F}$ such that $N - Q$ components, F_n, are zero (and therefore the corresponding complex amplitudes are also zero). This choice is highlighted in the summation Σ_n with a '*'. This enables us to limit the summation over modes to the same number as the exponent of the nonlinear coupling term.

We now seek one particular solution of equation (15) by singling out in the power expansion of equation (16) the term which is a product of Q different $(E_{n,j+1} - E_{n,j})(A_n e^{ik_n x} + A_n' e^{ik'_n x})$. That is, we define that term as:

$$(U_{j+1} - U_j)_s^Q = Q!\prod_n^* (A_n e^{ik_m x} + A_n' e^{ik'_n x})\prod_n^* (E_{n,j+1} - E_{n,j})e^{iQ\omega_d t} \tag{17}$$

The driving corresponding to this singled out term:

$$\vec{G}_s^{(0)} = Q!\prod_m^* (A_m e^{ik_n x} + A_m' e^{ik_n' x})\vec{g}^{(0)} e^{iQ\omega_d t} \tag{18}$$

with

$$g_1^{(0)} = \prod_n^* (E_{n,2} - E_{n,1}) \tag{19a}$$

$$g_{1<j<N}^{(0)} = \prod_n^* (E_{n,j} - E_{n,j-1}) + \prod_n^* (E_{n,j+1} - E_{n,j}) \tag{19b}$$

$$g_N^{(0)} = \prod_n^* (E_{n,N} - E_{n,N-1}) \tag{19c}$$

Again since the $\vec{E}_n$'s form a complete orthonormal basis, it is possible to expand $\vec{g}^{(0)}$ on that basis. For this we write:

$$\vec{g}^{(0)} = \sum_n \tilde{g}_n^{(0)}\vec{E}_n \tag{20}$$

To first order in perturbation, a particular solution, corresponding to the conditions above, of the nonlinear equation is solution of:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t}\right)\vec{I} + \alpha^2\vec{M}\right]\vec{U}_s^{(1)} = -Q!\prod_m^* (A_m e^{ik_m x} + A_m' e^{ik_m' x})\sum_n \tilde{g}_n^{(0)}\vec{E}_n e^{iQ\omega_d t} \tag{21}$$

Expanding the first order particular solution on the E_n basis:

$$\vec{U}_s^{(1)} = \sum_n u_n^{(1)}\vec{E}_n \tag{22}$$

And inserting equation (22) into equation (21) and using equation (2), equating the coefficients for each eigenvector, $\vec{E}_n$, results in the set of N equations:

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial t} + \alpha^2\lambda_n\right)u_n^{(1)} = -Q!\prod_m^* (A_m e^{ik_m x} + A_m' e^{ik_m' x})\tilde{g}_n^{(0)} e^{iQ\omega_d t} \tag{23}$$

To solve equation (23), we rewrite the product on the right hand side of the equal sign into the sum:

$$\prod_m^* (A_m e^{ik_m x} + A_m' e^{ik_m' x}) = \sum\nolimits'_{k_1,k_2,\ldots,k_Q} A_{k_1}A_{k_2}\ldots A_{k_Q}\varphi_{k_1,k_2\ldots k_Q} \tag{24}$$

where Σ' sums over all combinations of primed and unprimed k_m's of the allowed Q wave numbers. Here, $\varphi_{k_1,k_2\ldots k_Q} = e^{i(k_1+k_2+\ldots+k_Q)x}$. The complex amplitudes also take two possible values (primed and unprimed) for each of the Q allowable states. There are 2^Q terms in the summation of equation (24). We simplify the notation by reformulating the summation as $\sum_{l=1}^{2^Q} A_l \varphi_l$. We subsequently seek solutions in the form of the expansion:

$$u_n^{(1)} = \left(\sum_{l=1}^{2^Q} a_{l,n}\varphi_l\right) e^{iQ\omega_d t} \tag{25}$$

Inserting equation (25) into equation (23) and equating terms with the same phase φ_l gives:

$$a_{l,n} = -Q!\tilde{g}_n^{(0)} \frac{A_l}{D_{l,n}} \tag{26}$$

In equation (26), the denominator is expressed in the form:

$$D_{l,n} = -Q^2\omega_d^2 + \beta^2 K_l^2 + i\mu Q\omega_d + \alpha^2\lambda_n \tag{27}$$

The combination of indices $1, 2, \ldots, Q$ is that that correspond to the overall index l with K_l being the lth term in the sum of the left-hand side of equation (24).

In summary, the first order particular solution for the displacement is with all conventions introduced above:

$$\vec{U}_s^{(1)} = -Q!\sum_n \left(\sum_{l=1}^{2^Q} \frac{A_l}{D_{l,n}}\varphi_l\right) \tilde{g}_n^{(0)} \vec{E}_n e^{iQ\omega_d t} \tag{28}$$

Equation (28) can be reorganized as a linear combination of product states of plane waves:

$$\vec{U}_s^{(1)} = -Q!\sum_{l=1}^{2^Q} \left(\sum_n \frac{A_l}{D_{l,n}} \tilde{g}_n^{(0)} \vec{E}_n\right) \varphi_l e^{iQ\omega_d t} \tag{29}$$

4. Examples

The displacement field given by equation (29) is illustrated by three examples, namely $N = 3$ and $Q = 2$, $N = 4$ and $Q = 3$ and $N = 5$ and $Q = 4$.

4.1. Case $N = 3$ and $Q = 2$

In the case of three waveguides coupled in a planar array, the coupling matrix $\overleftrightarrow{M}$ admits three eigen vectors and three eigen values. One of the eigen vector for which $\lambda_1 = 0$, namely $\vec{E}_1^T = \frac{1}{\sqrt{3}}(1, 1, 1)$, does not involve transfer of energy between the waveguides via the coupling. This trivial case is equivalent to three independent waveguides. The other two eigen modes of the coupling matrix with eigen values $\lambda_2 = 1$, and $\lambda_3 = 3$, are:

$$\vec{E}_2 = \begin{pmatrix} E_{2,1} \\ E_{2,2} \\ E_{2,3} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{E}_3 = \begin{pmatrix} E_{3,1} \\ E_{3,2} \\ E_{3,3} \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{30}$$

We now suppose that we are driving the system with $F_1 = 0$ (to eliminate the $\vec{E}_1$ mode) and $F_2 \neq F_3 \neq 0$. The vector $\vec{g}^{(0)}$ has three components given by equations (19*a*)–(19*c*), namely:

$$\vec{g}^{(0)} = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \sqrt{\frac{3}{2}}\vec{E}_2 \tag{31}$$

According to equation (2), we therefore have $\tilde{g}_1^{(0)} = 0$, $\tilde{g}_2^{(0)} = \sqrt{\frac{3}{2}}$, and $\tilde{g}_3^{(0)} = 0$. The summation Σ_n in equation (29) has, therefore, only one non-zero term corresponding to $n = 2$. The summation $\Sigma_l^{2^{Q=2}}$ has four terms. The four denominators

$$D_{l,2} = -Q^2\omega_d^2 + \beta^2 K_l^2 + i\mu Q\omega_d + \alpha^2\lambda_2 \tag{32}$$

with $\lambda_2 = 1$, and K_l, A_l, $\varphi_l = [k_2 + k_3, A_2(k_2)A_3(k_3), e^{i(k_2+k_3)x}]$, $[k_2 + k_3', A_2(k_2)A_3(k_3'), e^{i(k_2+k_3')x}]$, $[k_2' + k_3, A_2(k_2')A_3(k_3), e^{i(k_2'+k_3)x}]$, $[k_2' + k_3', A_2(k_2')A_3(k_3'), e^{i(k_2'+k_3')x}]$ for $l = 1, 2, 3, 4$, respectively.

Here, the four numerators, A_l, and plane wave terms, φ_l, are those of equation (25) and the $A_n(k_n)$ are given by equation (7). The states of the system span a four dimensional Hilbert space product of two two dimensional Hilbert spaces.

For the sake of illustration, we will write explicitly only for that case all the terms contributing to the displacement field:

$$\vec{U}_s^{(1)} = -2!\sqrt{\frac{3}{2}}\vec{E}_2\left(\frac{A_2(k_2)A_3(k_3)}{-4\omega_d^2 + \beta^2(k_2 + k_3)^2 + 2i\mu\omega_d + \alpha^2}e^{i(k_2+k_3)x}\right.$$
$$+ \frac{A_2(k_2)A_3(k_3')}{-4\omega_d^2 + \beta^2(k_2 + k_3')^2 + 2i\mu\omega_d + \alpha^2}e^{i(k_2+k_3')x}$$
$$+ \frac{A_2(k_2')A_3(k_3)}{-4\omega_d^2 + \beta^2(k_2' + k_3)^2 + 2i\mu\omega_d + \alpha^2}e^{i(k_2'+k_3)x}$$
$$\left. + \frac{A_2(k_2')A_3(k_3')}{-4\omega_d^2 + \beta^2(k_2' + k_3')^2 + 2i\mu\omega_d + \alpha^2}e^{i(k_2'+k_3')x}\right)e^{i2\omega_d t} \tag{33}$$

Depending on the value of the complex coefficients of the amplitude of this superposition of states in the product space of plane waves, the superposition may be separable or nonseparable. It is nonseparable if it cannot be written in the form:

$$\vec{U}_{s,amp}^{(1)} = (\rho_2 e^{ik_2x} + \rho_2' e^{ik_2'x})(\tau_3 e^{ik_3x} + \tau_3' e^{ik_3'x}) \tag{34}$$

Here, the subscript *amp* refers to the amplitude part of equation (33). The degree of separability of the amplitude in equation (33) will be quantified in section 4 by calculating its entropy of 'entanglement.'

4.2. Case $N = 4$ and $Q = 3$

In the case of four waveguides coupled in a planar array, the coupling matrix $\overleftrightarrow{M}$ admits four eigen vectors and their corresponding eigen values. One of the eigen vector for which $\lambda_1 = 0$, namely $\vec{E}_1^T = \frac{1}{\sqrt{4}}(1, 1, 1, 1)$, again does not involve transfer of energy between the waveguides via the coupling. This trivial case is equivalent to four independent waveguides. The other three eigen modes of the coupling matrix with eigen values $\lambda_2 = 2 - \sqrt{2}, \ \lambda_3 = 2, \ \text{and } \lambda_4 = 2 + \sqrt{2}$, are:

$$\vec{E}_2 = \begin{pmatrix} E_{2,1} \\ E_{2,2} \\ E_{2,3} \\ E_{2,4} \end{pmatrix} = \frac{1}{2\sqrt{(2+\sqrt{2})}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \\ -1 \\ -1-\sqrt{2} \end{pmatrix}, \vec{E}_3 = \begin{pmatrix} E_{3,1} \\ E_{3,2} \\ E_{3,3} \\ E_{3,4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$
$$\vec{E}_4 = \begin{pmatrix} E_{4,1} \\ E_{4,2} \\ E_{4,3} \\ E_{4,4} \end{pmatrix} = \frac{1}{2\sqrt{(2+\sqrt{2})}}\begin{pmatrix} 1 \\ -1-\sqrt{2} \\ 1+\sqrt{2} \\ -1 \end{pmatrix} \tag{35}$$

We now suppose that we are driving the system with $F_1 = 0$ (to eliminate the $\vec{E}_1$ mode) and $F_2 \neq F_3 \neq F_4 \neq 0$. The vector $\vec{g}^{(0)}$ has four components given by equations (19a)–(19c), namely:

$$\vec{g}^{(0)} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\vec{E}_3 \tag{36}$$

According to equation (2), we therefore have for this particular case $\tilde{g}_1^{(0)} = 0$, $\tilde{g}_2^{(0)} = 0$, $\tilde{g}_3^{(0)} = \frac{1}{\sqrt{2}}$ and $\tilde{g}_4^{(0)} = 0$. The summation Σ_n in equation (29) has, therefore, only one non-zero term corresponding to $n = 3$. The summation $\Sigma_l^{2^{Q=3}}$ has eight terms. The eight denominators

$$D_{l,2} = -9\omega_d^2 + \beta^2 K_l^2 + 3i\mu\omega_d + \alpha^2\lambda_3 \tag{37}$$

with

$$K = \begin{pmatrix} k_2 + k_3 + k_4 \\ k_2 + k_3 + k_4' \\ k_2 + k_3' + k_4 \\ k_2' + k_3 + k_4 \\ k_2 + k_3' + k_4' \\ k_2' + k_3 + k_4' \\ k_2' + k_3' + k_4 \\ k_2' + k_3' + k_4' \end{pmatrix}; \ A = \begin{pmatrix} A_2(k_2)A_3(k_3)A_4(k_4) \\ A_2(k_2)A_3(k_3)A_4(k_4') \\ A_2(k_2)A_3(k_3')A_4(k_4) \\ A_2(k_2')A_3(k_3)A_4(k_4) \\ A_2(k_2)A_3(k_3')A_4(k_4') \\ A_2(k_2')A_3(k_3)A_4(k_4') \\ A_2(k_2')A_3(k_3')A_4(k_4) \\ A_2(k_2')A_3(k_3')A_4(k_4') \end{pmatrix}; \ \varphi = \begin{pmatrix} e^{i(k_2+k_3+k_4)x} \\ e^{i(k_2+k_3+k_4')x} \\ e^{i(k_2+k_3'+k_4)x} \\ e^{i(k_2'+k_3+k_4)x} \\ e^{i(k_2+k_3'+k_4')x} \\ e^{i(k_2'+k_3+k_4')x} \\ e^{i(k_2'+k_3'+k_4)x} \\ e^{i(k_2'+k_3'+k_4')x} \end{pmatrix} \tag{38}$$

Here, the particular first order states span an eight dimensional Hilbert space. This space is the product of three two dimensional Hilbert spaces.

4.3. Case $N = 5$ and $Q = 4$

In the case of five waveguides coupled in a planar array, the coupling matrix $\overleftrightarrow{M}$ admits five eigen vectors and their corresponding eigen values. One of the eigen vector for which $\lambda_1 = 0$, namely $\vec{E}_1^T = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)$, again is equivalent to five independent waveguides. The other four eigen modes of the coupling matrix with eigen values $\lambda_2 = \frac{1}{2}(3 - \sqrt{5}),\ \lambda_3 = \frac{1}{2}(5 - \sqrt{5}),\ \lambda_4 = \frac{1}{2}(3 + \sqrt{5})$ and $\lambda_5 = \frac{1}{2}(5 + \sqrt{5})$, are:

$$\vec{E}_2 = \begin{pmatrix} E_{2,1} \\ E_{2,2} \\ E_{2,3} \\ E_{2,4} \\ E_{2,5} \end{pmatrix} = \frac{1}{\sqrt{(5 - \sqrt{5})}} \begin{pmatrix} -1 \\ \frac{1}{2}(1 - \sqrt{5}) \\ 0 \\ -\frac{1}{2}(1 - \sqrt{5}) \\ 1 \end{pmatrix}, \vec{E}_3 = \begin{pmatrix} E_{3,1} \\ E_{3,2} \\ E_{3,3} \\ E_{3,4} \\ E_{3,5} \end{pmatrix} = \frac{1}{\sqrt{5(3 - \sqrt{5})}} \begin{pmatrix} 1 \\ \frac{1}{2}(-3 + \sqrt{5}) \\ 1 - \sqrt{5} \\ \frac{1}{2}(-3 + \sqrt{5}) \\ 1 \end{pmatrix},$$

$$\vec{E}_4 = \begin{pmatrix} E_{4,1} \\ E_{4,2} \\ E_{4,3} \\ E_{4,4} \\ E_{4,5} \end{pmatrix} = \frac{1}{\sqrt{(5 + \sqrt{5})}} \begin{pmatrix} -1 \\ \frac{1}{2}(1 + \sqrt{5}) \\ 0 \\ -\frac{1}{2}(1 + \sqrt{5}) \\ 1 \end{pmatrix}, \vec{E}_5 = \begin{pmatrix} E_{5,1} \\ E_{5,2} \\ E_{5,3} \\ E_{5,4} \\ E_{5,5} \end{pmatrix} = \frac{1}{\sqrt{5(3 + \sqrt{5})}} \begin{pmatrix} 1 \\ \frac{-1}{2}(3 + \sqrt{5}) \\ 1 + \sqrt{5} \\ \frac{-1}{2}(3 + \sqrt{5}) \\ 1 \end{pmatrix} \tag{39}$$

We now suppose that we are driving the system with $F_1 = 0$ (to eliminate the $\vec{E}_1$ mode) and $F_2 \neq F_3 \neq F_4 \neq F_5 \neq 0$. The vector $\vec{g}^{(0)}$ has five components given by equations (19a)–(19c), namely:

$$\vec{g}^{(0)} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 + \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 1 + \frac{1}{\sqrt{5}} \\ 1 \end{pmatrix} \tag{40}$$

In contrast to the two other previous cases, $\vec{g}^{(0)}$ is a linear combination of the spatial eigen vectors with $\tilde{g}_1^{(0)} = \frac{1}{\sqrt{5}}\left(1 + \frac{1}{\sqrt{5}}\right)$, $\tilde{g}_2^{(0)} = 0$, $\tilde{g}_3^{(0)} = \frac{-1}{4\sqrt{5(3 - \sqrt{5})}}\left(2 + \frac{1}{\sqrt{5}} - \sqrt{5}\right)$, $\tilde{g}_4^{(0)} = 0$ and $\tilde{g}_5^{(0)} = \frac{-1}{4\sqrt{5(3 - \sqrt{5})}}\left(\frac{1}{\sqrt{5}} + \sqrt{5}\right)$. In that case the particular solution for the displacement field becomes:

$$\vec{U}_s^{(1)} = -4! \sum_{l=1}^{2^4} A_l \left(\frac{1}{D_{l,1}} \tilde{g}_1^{(0)} \vec{E}_1 + \frac{1}{D_{l,3}} \tilde{g}_3^{(0)} \vec{E}_3 + \frac{1}{D_{l,5}} \tilde{g}_5^{(0)} \vec{E}_5\right) \varphi_l e^{i4\omega_d t} \tag{41}$$

with the 16 denominators for each n given by

$$D_{l,n} = -16\omega_d^2 + \beta^2 K_l^2 + i4\mu\omega_d + \alpha^2 \lambda_n \tag{42}$$

The 16 components of K_l, A_l, and φ_l are generalizations of those in equation (38). The Hilbert space supporting these states has dimension 2^4.

It is straight forward to generalize these examples to planar arrays of $N > 5$ waveguides and nonlinear exponent $Q = N - 1$. The dimension of the Hilbert space spanned by the nonlinear elastic states scales exponentially as 2^Q.

5. Entropy of entanglement

For the purpose of calculating the entropy of entanglement in the case of the example systems presented in the previous section and beginning with the first example $N = 3$ and $Q = 2$, we rewrite the complex coefficients in the superposition of product states as:

$$\vec{C}_l = \begin{pmatrix} C_{l,1} \\ C_{l,2} \\ C_{l,3} \end{pmatrix} = -2! \frac{A_l}{-4\omega_d^2 + \beta^2 K_l^2 + i2\mu\omega_d + \alpha^2 \lambda_2} \tilde{g}_2^{(0)} \vec{E}_2 \tag{43}$$

Let us now focus on the complex amplitude for one of the waveguides, say for example, the first waveguide, $C_{l,1}$. These amplitudes can be written in the form of a second rank tensor.

$$\boldsymbol{A}_1 = \begin{pmatrix} C_{1,1} & C_{2,1} \\ C_{3,1} & C_{4,1} \end{pmatrix} \tag{44}$$

Note that, by construction, the amplitude tensor $\overleftrightarrow{A}_1$ separates the k_2 and k_2' containing amplitudes by row, top and bottom, and the k_3 and k_3' containing amplitudes by column, top and bottom. The classical entanglement for this system is seen in the two dimensional Hilbert space of the product states represented by these amplitudes, hence we calculate the entanglement entropy from the amplitudes by forming the product $\boldsymbol{A}_1\boldsymbol{A}_1^\dagger$ which is proportional to the density matrix.

The product matrix is diagonalized using the transformation matrix $\boldsymbol{T}$ to give:

$$\boldsymbol{T}\boldsymbol{A}_1\boldsymbol{A}_1^\dagger T^\dagger = \begin{pmatrix} \sqrt{\sigma_+} & 0 \\ 0 & \sqrt{\sigma_-} \end{pmatrix} \tag{45}$$

$$\sigma_\pm = \frac{1}{2}(Tr(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger) \pm \sqrt{(Tr(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger))^2 - 4Det(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger)}) \tag{46}$$

Recalling that the product matrix is proportional to the density matrix, the entropy of entanglement is calculated as the von Neumann entropy with a normalization constraint:

$$S = -(\hat{\Sigma}_1^2 \ln \hat{\Sigma}_1^2 + \hat{\Sigma}_2^2 \ln \hat{\Sigma}_2^2) \tag{47}$$

Where $\frac{\sqrt{\sigma_+}}{\sqrt{\sigma_-}} = \frac{\hat{\Sigma}_1}{\hat{\Sigma}_2}$ and the normalized eigen values obey $\hat{\Sigma}_1^2 + \hat{\Sigma}_2^2 = 1$.

The entropy of entanglement can be rewritten in the form:

$$S = \ln 2 - \frac{1}{2}(1 - \sqrt{1 - 4R})\ln(1 - \sqrt{1 - 4R}) - \frac{1}{2}(1 + \sqrt{1 - 4R})\ln(1 + \sqrt{1 - 4R}) \tag{48}$$

In equation (48), $R = \frac{Det(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger)}{(Tr(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger))^2}$.

For a given finite length of the waveguides (i.e., discrete set of wavenumbers), it might be possible to choose the driving frequency, ω_d, such that the denominators of the diagonal terms are significantly smaller than those of the off diagonal terms. Let us also assume that the driving frequency is such that the numerators of the diagonal terms are nearly the same. In that case, if we neglect the off diagonal terms compared to the diagonal ones, then $(Tr(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger))^2\tilde{}4Det(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger)$ and the ratio $R\tilde{}1$. Under these conditions, $S = \ln 2$, and the system is maximally entangled. This system is equivalent to a maximally entangled Bell state. On the other hand, if the four denominator were the same (in the current situation this is highly improbable unless the wave numbers are the same), $Det(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger) = 0$ i.e., $R = 0$ and subsequently $S = 0$. The superposition would be separable.

Extending this discussion to the $\mathrm{N} = 4$ and $\mathrm{Q} = 3$ system, recall the complex coefficients in the superposition of product states:

$$\vec{C}_l = \begin{pmatrix} C_{l,1} \\ C_{l,2} \\ C_{l,3} \\ C_{l,4} \end{pmatrix} = -3!\frac{A_l}{-4\omega_d^2 + \beta^2 K_l^2 + i3\mu\omega_d + \alpha^2\lambda_2}\tilde{g}_3^{(0)}\vec{E}_3 \tag{49}$$

The coefficients can be rearranged in the form of an unfolding tensor

$$\boldsymbol{A}_1 = \begin{pmatrix} C_{1,1} & C_{3,1} & C_{2,1} & C_{5,1} \\ C_{3,1} & C_{7,1} & C_{6,1} & C_{8,1} \end{pmatrix} \tag{50}$$

The amplitude matrix for this system is 4×2 and is ordered such that the 2×2 matrix on the left has amplitudes including $k_2, k_2', \ k_3, k_3'$, as in the previous system and amplitudes that include k_4, while the 2×2 matrix on the right has amplitudes that include k_4', see equation (38).

This construction can be seen as a bipartite system which spans an 8 dimensional Hilbert space tensor product of a two-dimensional Hilbert and a 4 dimensional Hilbert space. We employ the singular value decomposition (SVD) to factorize the 2×4 complex matrix, $\boldsymbol{A}_1$ into the product of a 2×2 complex unitary matrix, $\boldsymbol{X}$, a 2×4 rectangular diagonal matrix, $\boldsymbol{\Sigma}$, with singular real positive values on the diagonal and a 4×4 complex unitary matrix, $\boldsymbol{V}^\dagger$, where the dagger refers to the conjugate transpose matrix. The SVD is written as:

$$\boldsymbol{A}_1 = \boldsymbol{X}\ \boldsymbol{\Sigma}\ \boldsymbol{V}^\dagger \tag{51}$$

To find the singular values, we determine the eigen values of $\boldsymbol{A}_1\boldsymbol{A}_1^\dagger = \boldsymbol{X}\ \boldsymbol{\Sigma}\ \boldsymbol{\Sigma}^\dagger\boldsymbol{X}^\dagger$. Since $\underline{\boldsymbol{X}}$ is unitary, the square root of the eigen values of $\boldsymbol{A}_1\boldsymbol{A}_1^\dagger$ will be the singular values of $\boldsymbol{\Sigma}$. Note that the SVD is a generalization of the diagonalization employed for the $N = 3$ & $Q = 2$ system.

The result takes on again the general form of equation (46) with the two eigen values given by:

$$\sigma_\pm = \frac{1}{2}(Tr(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger) \pm \sqrt{(Tr(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger))^2 - 4Det(\boldsymbol{A}_1\boldsymbol{A}_1^\dagger)}) \tag{52}$$

The rectangular diagonal matrix, Σ, then is obtained as:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \Sigma_1 & 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_+} & 0 & 0 & 0 \\ 0 & \sqrt{\sigma_-} & 0 & 0 \end{pmatrix} \tag{53}$$

Note that the matrix $\boldsymbol{A}_1\boldsymbol{A}_1^\dagger$ is a 2×2 matrix and the analysis of the entanglement entropy follows the development for the previous system.

The case of the $N = 5$ and $Q = 4$, the amplitude matrix is 2×8 and can be constructed by replicating the 2×4 matrix of the previous system i.e., $N = 4$ and $Q = 3$ on left and right, segregating terms in the left and right 2×4 matrices as before while writing amplitudes that depend on k_5 in the right 2×4 matrix and those depending on k_5' in the left. SVD can be used to generate the entropy of entanglement as in equations (52) and (53) for any planar array of N nonlinearly coupled waveguides with nonlinear exponent $Q < N$.

6. Conclusions

We have shown that we can achieve exponentially complex superpositions of product states in externally driven planar arrays of N one-dimensional elastic waveguides coupled nonlinearly along their length. The nonlinear coupling is a power function of the relative displacements between waveguides with exponent Q. Such systems can be realized experimentally by binding near one-dimensional elastic rods with intrinsically nonlinear elastic media to form the parallel arrays of waveguides. Other approach may use granular materials as coupling agent between the waveguides/rods to exploit Hertzian contact as a source of nonlinearity.

To first order in perturbation, if we excite Q of the N possible elastic bands and two plane wave states in each band, the nonlinear elastic system can be visualized as a multipartite two level system which can support superpositions of nonlinear modes which span exponentially complex Hilbert spaces of dimension 2^Q. By partitioning the spatial degrees of freedom across the planar array, i.e., separating the system into two subsystems, we introduce and implement an approach using singular value decompositions to calculate the entropy of entanglement from the density matrix. We show the possibility of achieving maximally entangled exponentially complex superpositions of nonlinear product states. The exponentially complex Hilbert space can be navigated by varying the driving frequency and the amplitude of the drivers. The nonseparable superpositions of product states can be measured experimentally using contact and/or noncontact probes such as arrays of transducers to determine the relative amplitude and phase of the displacement field in the waveguides to get access to the $\vec{E}_n$ states. Laser Doppler vibrometry can be used to scan the elastic field along the waveguides. By spatial Fourier transformation of the displacement field along the waveguides measured via noncontact laser vibrometry, we can access the exponentially complex wave number space of K.

This work expands the capability of arrays of nonlinearly coupled elastic waveguides to produce quantum like phenomena. Nonseparable superpositions of elastic product states are not subject to the instability due to decoherence of true quantum systems. Recently, the use of classical light with entangled degrees of freedom have found applications in quantum information [32] and metrology [33, 34]. The present study suggests that the same sort of applications can be realized with acoustic systems. Moreover, due to the flexibility of elastic system, the coupling between elastic waveguides can easily be tailored to be linear, periodic, and nonlinear (with different types or degrees of nonlinearity) [35, 36], which in turn would widen the Hilbert space accessible to elastic superposition of states. This coupling can be easily manipulated by choices of materials and fabrication. Finally, externally driven arrays of nonlinearly coupled elastic waveguides, with their ability to navigate a large section of an exponentially complex Hilbert space of product of plane wave states, may therefore offer a stable classical alternative to quantum bits for massive information storage or processing.

Acknowledgments

We acknowledge support from the W M Keck foundation.

ORCID iDs

P A Deymier https://orcid.org/0000-0002-1088-7958
K Runge https://orcid.org/0000-0001-6467-7481
M Arif Hasan https://orcid.org/0000-0001-7452-004X

References

[1] Dowling J P and Milburn G J 2003 Quantum technology: the second quantum revolution *Philos. Trans. R. Soc. London. Ser. Math. Phys. Eng. Sci.* **361** 1655–74
[2] Spreeuw R J C 1998 A classical analogy of entanglement *Found. Phys.* **28** 361–74
[3] Ghose P and Mukherjee A 2014 Entanglement in classical optics *Rev. Theor. Sci.* **2** 274–88
[4] Karimi E and Boyd R W 2015 Classical entanglement? *Science* **350** 1172–3
[5] Souza C E R, Huguenin J A O, Milman P and Khoury A Z 2007 Topological phase for spin–orbit transformations on a laser beam *Phys. Rev. Lett.* **99** 160401
[6] Chen L L and She W 2010 Single-photon spin–orbit entanglement violating a Bell-like inequality *J. Opt. Soc. Am.* B **27** A7
[7] Borges C V S, Hor-Meyll M, Huguenin J A O and Khoury A Z 2010 Bell-like inequality for the spin–orbit separability of a laser beam *Phys. Rev.* A **82** 033833
[8] Karimi E, Leach J, Slussarenko S, Piccirillo B, Marrucci L, Chen L, She W, Franke-Arnold S, Padgett M J and Santamato E 2010 Spin–orbit hybrid entanglement of photons and quantum contextuality *Phys. Rev.* A **82** 022115
[9] Valles A, D'Ambrosio V, Hendrych M, Micuda M, Marrucci L, Sciarrino F and Torres J P 2014 Generation of tunable entanglement and violation of a Bell-like inequality between different degrees of freedom of a single photon *Phys. Rev.* A **90** 052326
[10] Pereira L J, Khoury A Z and Dechoum K 2014 Quantum and classical separability of spin–orbit laser modes *Phys. Rev.* A **90** 053842
[11] Qian X-F, Little B, Howell J C and Eberly J H 2015 Shifting the quantum–classical boundary: theory and experiment for statistically classical optical fields *Optica* **2** 611
[12] Balthazar W F, Souza C E R, Caetano D P, Galvão E F, Huguenin J A O and Khoury A Z 2016 Tripartite nonseparability in classical optics *Opt. Lett.* **41** 5797
[13] Rafsanjani S M H, Mirhosseini M, Magana-Loaiza O S and Boyd R W 2015 State transfer based on classical nonseparability *Phys. Rev.* A **92** 023827
[14] Michler M, Weinfurter H and Żukowski M 2000 Experiments towards falsification of noncontextual hidden variable theories *Phys. Rev. Lett.* **84** 5457
[15] Gadway B, Galvez E and Zela F D 2008 Bell-inequality violations with single photons entangled in momentum and polarization *J. Phys. B: At. Mol. Opt. Phys.* **42** 015503
[16] Hasegawa Y, Loidl R, Badurek G, Baron M and Rauch H 2003 Violation of a Bell-like inequality in single-neutron interferometry *Nature* **425** 45
[17] Deymier P and Runge K 2017 *Sound Topology, Duality, Coherence and Wave-Mixing: An Introduction to the Emerging New Science of Sound* (Springer International Publishing)
[18] Zhang Z, Wei Q, Cheng Y, Zhang T, Wu D and Liu X 2017 Topological creation of acoustic pseudospin multipoles in a flow-free symmetry-broken metamaterial lattice *Phys. Rev. Lett.* **118** 084303
[19] Zhang Z, Tian Y, Cheng Y, Liu X and Christensen J 2017 Experimental verification of acoustic pseudospin multipoles in a symmetry-broken snowflakelike topological insulator *Phys. Rev.* B **96** 241306
[20] Long Y, Ren J and Chen H 2018 Intrinsic spin of elastic waves *PNAS* **115** 9951–5
[21] Calderin L, Hasan M A, Jenkins N G, Lata T, Lucas P, Runge K and Deymier P A 2019 Experimental demonstration of coherent superpositions in an ultrasonic pseudospin *Sci. Rep.* **9** 1–10
[22] Li J, Wang J, Wu S and Mei J 2017 Pseudospins and topological edge states in elastic shear waves *AIP Adv.* **7** 125030
[23] Wang L, Zhang R-Y, Xiao M, Han D, Chan C T and Wen W 2016 The existence of topological edge states in honeycomb plasmonic lattices *New J. Phys.* **18** 103029
[24] Pal R K and Ruzzene M 2017 Edge waves in plates with resonators: an elastic analogue of the quantum valley hall effect *New J. Phys.* **19** 025001
[25] Hasan M A, Calderin L, Lucas P, Runge K and Deymier P A 2019 Geometric phase invariance in spatiotemporal modulated elastic system *J. Sound Vib.* **459** 114843
[26] Hasan M A, Calderin L, Lucas P, Runge K and Deymier P A 2019 Spectral analysis of amplitudes and phases of elastic waves: application to topological elasticity *The Journal of the Acoustical Society of America* **146** 748–66
[27] Wang H-X, Guo G-Y and Jiang J-H 2019 Band topology in classical waves: wilson-loop approach to topological numbers and fragile topology *New J. Phys.* **21** 093029
[28] Zhang D, Ren J, Zhou T and Li B 2019 Dark state, zero-index and topology in phononic metamaterials with negative mass and negative coupling *New J. Phys.* **21** 093033
[29] Meng Y *et al* 2018 Designing topological interface states in phononic crystals based on the full phase diagrams *New J. Phys.* **20** 073032
[30] Hasan M A, Calderin L, Lata T, Lucas P, Runge K and Deymier P A 2019 The sound of Bell states *Communication Physics* **2** 1
[31] Deymier P A, Runge K, Hasan M A and Calderin L 2019 Exponentially complex 'classically entangled' states in arrays of one-dimensional nonlinear elastic waveguides *Materials* **12** **3553**
[32] Gabriel C *et al* 2011 Entangling different degrees of freedom by quadrature squeezing cylindrically polarized modes *Phys. Rev. Lett.* **106** 060502
[33] Töppel F, Aiello A, Marquardt C, Giacobino E and Leuchs G 2014 Classical entanglement in polarization metrology *New J. Phys.* **16** 073019
[34] Simon B N, Simon S, Gori F, Santarsiero M, Borghi R, Mukunda N and Simon R 2010 Nonquantum entanglement resolves a basic issue in polarization optics *Phys. Rev. Lett.* **104** 023901
[35] Chen Q and Elbanna A 2017 Emergent wave phenomena in coupled elastic bars: from extreme attenuation to realization of elastodynamic switches *Sci. Rep.* **7** 1
[36] Hikihara T, Okamoto Y and Ueda Y 1997 An experimental spatio-temporal state transition of coupled magneto-elastic system *Chaos* **7** 810